THE MATHEMATICAL SOLUTION OF ENGINEERING PROBLEMS

The Mathematical Solution of Engineering Problems

BY MARIO G. SALVADORI

ASSOCIATE PROFESSOR OF CIVIL ENGINEERING,
COLUMBIA UNIVERSITY

With a Collection of Problems by

KENNETH S. MILLER

NEW YORK TORONTO LONDON
McGRAW-HILL BOOK COMPANY, INC.
1948

THE MATHEMATICAL SOLUTION OF ENGINEERING PROBLEMS

To

G. *and* V.

PREFACE

This book has evolved from a set of mimeographed lecture notes used in a one-semester course in engineering mathematics offered by the senior author in the School of Engineering of Columbia University for the last seven years. The course is the first of a sequence of four aimed at widening the mathematical background of undergraduate and graduate students and at filling the gap between a knowledge of theoretical mathematics and the technique of solving physical problems by mathematical methods.

It is well known that the student's interest in a subject is greatly enhanced whenever he is shown beforehand its importance in connection with his field of work. In this elementary text a simple physical problem first motivates the introduction of each mathematical technique and of the corresponding theory; the solution of the problem is then carried to its final numerical result, with special emphasis on accuracy of computation and practical numerical schemes. Proofs are mentioned only to develop the treatment of the material in a logical manner.

Although the reader is assumed to have a knowledge of college mathematics, including the calculus, it has been found extremely useful to devote about one-third of the book to a review, from an intuitional standpoint, of those fundamental concepts of mathematics which are so often forgotten by the student after his first rapid excursion in the field. This review, on the one hand, clarifies the basic ideas repeatedly used in the applications and, on the other, constitutes a reminder of the elementary techniques learned in pure mathematics courses.

The topics treated have been chosen by polling members of the staff of the School as to the mathematical needs of their courses. It was found that the same topics are also of interest to practicing engineers requiring a review course on mathematics or a wider mathematical background in connection with their work, as well as to students in mathematics, chemistry, and physics wishing to become acquainted with the engineering approach to physical problems. Hence the book is intended for use by readers with various technical backgrounds.

The six chapters of the book deal with

1. The fundamental ideas of mathematics (number, variable, func-

tion, limit, continuity, infinitesimal, derivative, differential, and integral) and related techniques.
2. The use of plane Cartesian geometry.
3. The solution of algebraic and transcendental equations.
4. The solution of systems of simultaneous linear equations.
5. The elementary functions of a real and of a complex variable and power series expansion.
6. The Fourier series expansion and harmonic analysis.

The illustrative problems used in the text are taken from elementary physics and mechanics and from the various branches of engineering, but their understanding does not require familiarity with senior-year subjects.

In any book trying to teach mathematical techniques the sections on problems are among the most important. At the end of each chapter the reader will find a series of problems, which may be divided into two categories: practice problems and applied problems. The first type of exercise is aimed at improving the mathematical skill of the student and does not require a knowledge of either physics or engineering; the second is aimed at teaching the reader how to formulate and solve physical problems in mathematical terms. Answers to alternate problems are given at the end of the book.

Kenneth S. Miller, after giving considerable help in the preparation of the manuscript, undertook the strenuous job of gathering and solving the more than one thousand problems contained in the book. I am glad to take this opportunity to express my appreciation for his efforts.

I fulfill a debt of gratitude to the memory of the late Prof. G. B. Karelitz by remembering his help in starting an engineering mathematics course at Columbia, and I am grateful to many colleagues in the School for their encouragement.

To my friend Prof. R. D. Mindlin I am particularly indebted for his continued interest in my work and his illuminating advice.

I also wish to express my thanks to Miss W. M. Curtis and Miss E. J. Johnson for their skillful and untiring efforts in the preparation of the manuscript, and finally to my numerous students, who have contributed so much to my interest and knowledge in the field of engineering mathematics by their challenging questions.

MARIO G. SALVADORI

NEW YORK, N.Y.
July, 1948

CONTENTS

INTRODUCTION

Mathematics is a type of shorthand particularly well adapted to the language of logic or common sense. Common sense, like any other faculty, can be improved only by practice. Mathematical common sense will be improved by using mathematics, but this can be done only by first mastering its shorthand alphabet.

By careful study of this short book the reader will first learn this alphabet and build up a mathematical vocabulary. He will then be shown how to use this basic language to extend the range of his mathematical ideas and to attack successfully a large variety of engineering problems.

To the engineer, mathematics is a tool. Far from despising the pioneering and creative effort of the pure mathematician, the engineer is compelled to take for granted most of the rigorous proofs of mathematical theorems and to concentrate on the question: "How can I apply these abstract truths to my problem?" For this reason the fundamental ideas of mathematics will be introduced to the reader from an intuitional point of view and will be clarified by means of their application to elementary engineering problems. This viewpoint also has the great advantage of motivating to the reader the study of certain mathematical ideas and techniques.

The reader should keep in mind that, in any book attempting to teach "how to do things," the section on problems is the most important. If and when he learns how to solve most of the problems presented at the end of each chapter, he will be well equipped to undertake mathematically elementary engineering problems. More difficult problems may require much more mathematics than can be covered in a single book, but advanced work will appear easy once the fundamentals have been thoroughly mastered.

CHAPTER I

A REVIEW OF SOME BASIC MATHEMATICAL CONCEPTS

1·1 Numbers

The ultimate result of engineering work is very often a number, which may be obtained after many computations and which must be correct lest all the labor involved in getting it be of no avail. It is only proper that we should know what kinds of numbers we shall meet in our computations.

Natural numbers, as the positive integers are sometimes called, are so deeply rooted in our minds that the German mathematician Kronecker said: "God made the integers, man the other numbers." Within the field of positive integers, the operation of addition is always possible, since the sum of two or more positive integers is always a positive integer. But the field of positive integers is insufficient to perform the operation of subtraction *in all cases.* Therefore, *negative integers* and *zero* have to be invented for this purpose. Geometrically minded people like the Greeks, to whom numbers very often meant distances, had no use for negative integers. The Hindus, on the other hand, who were great traders, knew that in business one can be "in the red" and used negative integers.

The operation of multiplication, a shorthand operation for addition, can always be performed within the field of integers. But the operation of division, in order always to be performable, requires an extended field of numbers called *fractions.*

In measuring the length of a segment we often use feet and inches, *i.e.,* fractions of 1 ft, in order to be more accurate. But it must be noticed that by using a smaller unit, say the inch, the measure of a length can always be represented by an integer. Thus a length of 2 ft 4 in. is represented mathematically by the fraction $^{28}/_{12}$ in feet and by the integer 28 in inches.

A fraction a/b is defined by the two integers a,b. This shows that a number of the new family, the fractions, is defined by an *ordered* couple of numbers of the previous family, the integers, since a/b is different from b/a. Once fractions have been introduced, we extend the rules of algebra to this new family in such a manner that the rules on integers (*i.e.,* on the special fractions a/b where $b = 1$) remain unchanged.

In engineering, most fractions are written in decimal form. Some fractions give rise to decimal numbers with a finite number of digits,

for example, $\frac{1}{4} = 0.25$; others are representable by decimals with an infinite number of digits, which will *always repeat in groups*, for example, $\frac{1}{7} = 0.142857\ 142857\ 142857\ \ldots$

If a unit length and an origin are chosen on a straight line (usually called the *x axis*), all the integers and all the fractions can be represented by points on the straight line, as shown in Fig. 1·1. The family of integers and fractions is called the field of *rational numbers*.

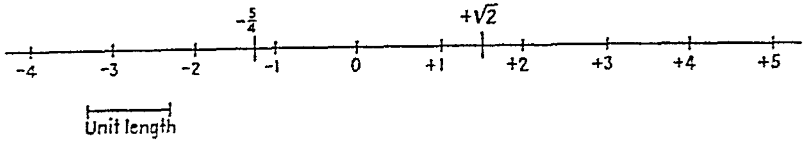


FIG. 1·1.

Let us now build, by compass and straightedge, the right triangle of sides equal to those of the triangle shown in Fig. 1·2, whose diagonal by Pythagoras's theorem has a length $L = \sqrt{1^2 + 1^2} = \sqrt{2}$. On the axis of Fig. 1·1 there is a point whose distance from the origin is equal to $\sqrt{2}$. But $\sqrt{2}$ is neither an integer nor a fraction. It is not an integer, since

$$1^2 < 2 < 2^2$$

and, taking the square root of each term of this inequality,

$$1 < \sqrt{2} < 2$$

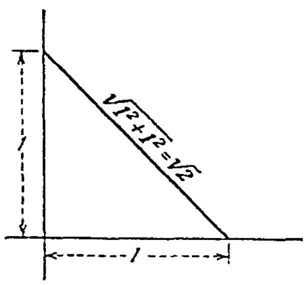


FIG. 1·2.

To prove that $\sqrt{2}$ is *not* a fraction, we assume instead that

$$\sqrt{2} = \frac{m}{n},$$

where m and n are two integers having no common factors. (If they have, we first divide them both by these common factors.) Squaring and multiplying by n^2, we obtain

$$2n^2 = m^2$$

This equality shows that m^2 is twice n^2 and that, hence, m^2 and n^2 must have common factors. But, since by assumption m and n have no common factors, m^2 and n^2 do not have common factors either. This contradiction proves that the assumption $\sqrt{2} = m/n$ is false, *i.e.*, that $\sqrt{2}$ is not a fraction.

It is thus seen that, while the operation of taking powers of a positive rational number can always be performed in the field of rational numbers,

the operation of taking roots of a positive rational number requires a wider field, the field of *irrational numbers,* to be always performable. Famous among irrational numbers are $\sqrt{2}$, $\sqrt{3}$, π, and e (base of Napierian logarithms). Irrational numbers, when written in decimal form, have an infinite number of decimal figures, which follow at random.

The rational and irrational numbers form the field of *real numbers,* which covers *all* the points of a straight line without gaps. The number 1 is called the *unit of real numbers.*

The square root of a negative number cannot be taken within the field of real numbers, since no real number gives a negative result when squared. But since one of the steps in many computations is found to involve the evaluation of such square roots, a new kind of number has been invented in order that the operation "take an even root" be performable on *all* real numbers. These new numbers are called *imaginaries.* The imaginary number $\sqrt{-4}$ is often written as

$$\sqrt{-4} = \sqrt{4(-1)} = \sqrt{4}\,\sqrt{-1} = 2\sqrt{-1} = 2i$$

calling i the new symbol $\sqrt{-1}$, which has *no meaning* in the field of real numbers. The symbol i is the *unit of imaginary* numbers and is an entirely different kind of unit from 1, the real unit. The only tie between i and 1 is that $i^2 = -1$. In other words, multiplication of one *kind* of number, say the real number 2, by i, changes it into a *different kind* of number, the imaginary $2i$, while a second multiplication by i gives again a real number, -2.[1]

Numbers of the real kind, like $a \cdot 1$, and of the imaginary kind, like $b \cdot i$, cannot be added, just as different kinds of objects, say potatoes and tomatoes, cannot be added. But it is not uncommon to find in engineering work that the result of a computation cannot be expressed by a real *or* an imaginary number alone; both a real *and* an imaginary number are required, as in the solution of the quadratic equation

$$x^2 + 2x + 2 = 0$$

whose roots are

$$x = \tfrac{1}{2}(-2 \pm \sqrt{4 - 8}) = -1 \pm i$$

It was noticed before that two integers are needed to define a fraction. Similarly, two real numbers (a,b) are needed to express the fact that the

[1] That the same operation may give essentially different results when repeated should not amaze the reader. Suppose that a man is very sick and is ordered by his doctor to take *one* tablet of a medicine containing a poisonous ingredient. He follows the prescription and gets well. The result of the operation "take one tablet" applied once is "life." If the man applies the same operation twice, *i.e.*, takes *two* pills, he is poisoned and dies. The result of applying twice in a row the lifesaving operation "take one tablet" is entirely different—it is "death."

answer of a problem is a units 1 and b units i, and it is convenient to consider this composite answer as a single number of a new kind, a *complex number*. A complex number z is therefore defined by the ordered couple (a,b) and means a units 1 *and* b units i. When $b = 0$, we get the special case of real numbers; when $a = 0$, the special case of imaginary numbers.

It is customary to write the complex number $z = (a,b)$ in the form

$$z = a + bi$$

but it cannot be overemphasized that the plus sign in this formula *does not stand for addition*. The only justification for the use of this notation is that the operations of addition, subtraction, and multiplication on complex numbers are so defined that the usual rules of the algebra of real numbers hold for complex numbers, if the plus sign is *formally* considered as an addition sign. Thus, given two complex numbers $(a,b) = a + bi$ and $(c,d) = c + di$,

$$(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i \tag{1·1·1}$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i \tag{1·1·2}$$

Two complex numbers are called *conjugate* when they have the same *real part* and opposite *imaginary parts*. The product of two conjugate numbers is always a positive real number,

$$(a + bi)(a - bi) = (a^2 + b^2) + (ab - ab)i = a^2 + b^2$$

whose positive square root is called the *modulus* of the complex number, or its *absolute value*, and is indicated by the symbol

$$|z| = +\sqrt{a^2 + b^2} \tag{1·1·3}$$

In the particular case of real numbers ($b = 0$) the absolute value of a number is its positive or *numerical* value. Thus

$$|4| = +\sqrt{4^2} = +4 \qquad |-3| = +\sqrt{(-3)^2} = +3$$

To divide one complex number by another, we multiply both by the conjugate of the divisor,

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\,i \tag{1·1·4}$$

Just as a real number is represented by a point on the x axis, a complex number is represented by a point in the x,y plane. The point is obtained

by taking as x coordinate the real part of the number and as y coordinate its imaginary part (Fig. 1·3). The modulus of the number is equal to the distance r of the point from the origin. Real numbers ($b = 0$) appear on the x axis, imaginaries ($a = 0$) on the y axis.[1] In more advanced work, numbers with more than two units are sometimes used. Quaternions have, for instance, four separate units, e_1,e_2,e_3, and e_4, and are written as

$$Z = ae_1 + be_2 + ce_3 + de_4$$

while space vectors have units i_x, i_y, and i_z and can be written as

$$V = xi_x + yi_y + zi_z$$

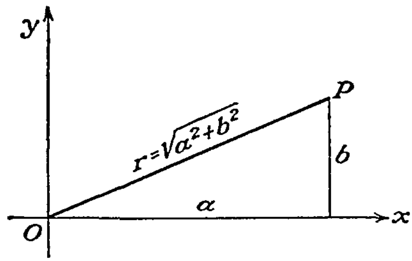


FIG. 1·3.

Table 1·1 shows the different kinds of numbers used in our computations and their relation to one another.

TABLE 1·1

$$\left.\begin{array}{l}\left.\begin{array}{l}\left.\begin{array}{l}\text{Natural numbers}\\ \text{Zero}\\ \text{Negative integers}\end{array}\right\}\text{Integers}\\ \text{Fractions}\end{array}\right\}\text{Rationals}\\ \text{Irrationals}\end{array}\right\}\text{Reals}$$

$$\left.\begin{array}{l}\text{Reals}\\ \text{Imaginaries}\end{array}\right\}\text{Complex numbers}$$

1·2 Operations on Complex Numbers

In order to multiply, divide, and take powers and roots of a complex number it is often convenient to locate the representative point in the x,y plane by means of polar coordinates (Fig. 1·4).

The polar coordinates of the point $P(x,y)$, representative of the complex number

$$z = x + yi \tag{1·2·1}$$

are given by

$$r = +\sqrt{x^2 + y^2} \tag{1·2·2}$$

$$\theta = \arctan \frac{y}{x} \tag{1·2·3}$$

where r, as indicated, is always positive.

Since, from Fig. 1·4,

$$x = r \cos \theta \qquad y = r \sin \theta$$

[1] In certain problems it is convenient to consider the number $x + iy$ as represented by the *vector* OP rather than by the point P.

the complex number z can be written in its *trigonometric form* as

$$z = r(\cos\theta + i\sin\theta) \tag{1·2·4}$$

or simply

$$z = r\underline{/\theta} \tag{1·2·4a}$$

Given another complex number in its trigonometric form,

$$z' = r'(\cos\theta' + i\sin\theta')$$

the product $z \cdot z'$ can be immediately obtained, remembering the formulas for the sine and cosine of the sum of two angles, in the form

$$\begin{aligned} z \cdot z' &= rr'[(\cos\theta\cos\theta' - \sin\theta\sin\theta') + i(\sin\theta\cos\theta' + \cos\theta\sin\theta')] \\ &= rr'[\cos(\theta + \theta') + i\sin(\theta + \theta')] \end{aligned} \tag{1·2·5}$$

Equation (1·2·5) proves that the modulus of the product is equal to the *product* of the moduli, while the angle (or phase) is equal to the *sum* of the angles.

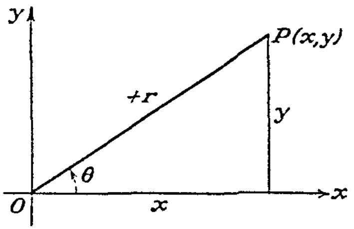


FIG. 1·4.

Similarly, the operation of division gives, calling

$$\bar{z}' = r'(\cos\theta' - i\sin\theta')$$

the conjugate of z' and remembering that $z' \cdot \bar{z}' = (r')^2$,

$$\begin{aligned} \frac{z'}{z} = \frac{z\bar{z}'}{z'\bar{z}'} &= \frac{rr'}{(r')^2}[(\cos\theta\cos\theta' + \sin\theta\sin\theta') + i(\sin\theta\cos\theta' - \sin\theta'\cos\theta)] \\ &= \frac{r}{r'}[\cos(\theta - \theta') + i\sin(\theta - \theta')] \end{aligned} \tag{1·2·6}$$

Thus the modulus of the *ratio* is equal to the *ratio* of the moduli, while the angle is equal to the *difference* of the angles.

If in Eq. (1·2·5) we take $z' = z$, that is, $r = r'$ and $\theta = \theta'$, we get the square of z,

$$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$$

Multiplying z^2 by z we get $z^3 = r^3(\cos 3\theta + i\sin 3\theta)$ and, in general, with n a positive integer,

$$z^n = r^n(\cos n\theta + i\sin n\theta) \tag{1·2·7}$$

Equation (1·2·7) (*de Moivre's formula*) can be proved to hold for any real number n and is used in computing the n separate nth roots of a complex number. Let us assume that the number

$$z = r(\cos\theta + i\sin\theta) \tag{1·2·8}$$

is one of the nth roots of the number

$$Z = R(\cos \varphi + i \sin \varphi)$$

This simply means that $z^n = Z$, that is, according to Eq. (1·2·7), that

$$z^n = r^n(\cos n\theta + i \sin n\theta) = R(\cos \varphi + i \sin \varphi) \tag{1·2·9}$$

Two complex numbers are equal if and only if their real parts *and* their imaginary parts are equal. Therefore Eq. (1·2·9) holds if and only if

$$\left.\begin{aligned} r^n \cos n\theta &= R \cos \varphi \\ r^n \sin n\theta &= R \sin \varphi \end{aligned}\right\} \tag{1·2·10}$$

Squaring and adding these two equations, we find that $(r^n)^2 = R^2$ and hence that $r^n = R$, or

$$r = \sqrt[n]{R} \tag{1·2·11}$$

i.e., the modulus of the nth root of a number is equal to the nth root of the modulus of the number (a real positive nth root of R always exists, since R is a positive number). Since $r^n = R$, Eqs. (1·2·10) show that the two angles $n\theta$ and φ must have the same cosine *and* the same sine. But two angles have the same cosine and the same sine when they are equal or differ by multiples of 360 deg, that is, of 2π radians. Hence

$$n\theta = \varphi + k2\pi \qquad (k \text{ an integer})$$

or

$$\theta = \frac{\varphi}{n} + k\frac{2\pi}{n} \quad (k = 0, 1, 2, \ldots, n-1) \tag{1·2·12}$$

Equation (1·2·12) gives n separate values of the angle θ, having different values of the sine and cosine, by means of which Eqs. (1·2·8) and (1·2·11) give the n roots of Z. [The angles obtained from Eq. (1·2·12) by making $k = n, n+1, n+2, \ldots$ have cosines and sines identical with those of the previous angles. Hence only n separate nth roots are obtained.]

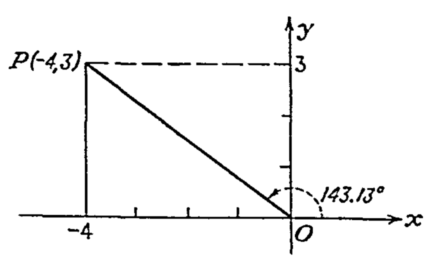


FIG. 1·5.

Let us compute, for example, the four 4th roots of the number

$$Z = -4 + 3i,$$

represented in Fig. 1·5 by the point P. By Eqs. (1·2·2) and (1·2·3),

$$R = \sqrt{4^2 + 3^2} = 5$$
$$\varphi = \arctan\left(-\tfrac{3}{4}\right) = 143.13°$$

By Eq. (1·2·11),

$$r = \sqrt[4]{5} = 1.495$$

and by Eq. (1·2·12)

$$\theta = \frac{143.13°}{4} + k\frac{360°}{4} \qquad (k = 0, 1, 2, 3)$$

or for

$$\begin{aligned} k &= 0 & \theta &= 35.78° \\ k &= 1 & \theta &= 125.78° \\ k &= 2 & \theta &= 215.78° \\ k &= 3 & \theta &= 305.78° \end{aligned}$$

Hence, by Eq. (1·2·8),

$$z_1 = 1.495(\cos 35.78° + i \sin 35.78°) = 1.495(0.8112 + i\,0.5845) = 1.213 + 0.8738i$$

$$z_2 = 1.495(\cos 125.78° + i \sin 125.78°) = 1.495(-0.5845 + i\,0.8112) = -0.8738 + 1.213i$$

$$z_3 = -1.213 - 0.8738i \qquad z_4 = 0.8738 - 1.213i$$

The same result can be obtained graphically by drawing a circle of radius $r = \sqrt[n]{R}$, by marking on its circumference the point P_1, which

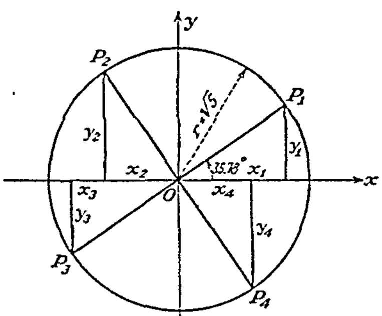


FIG. 1·6.

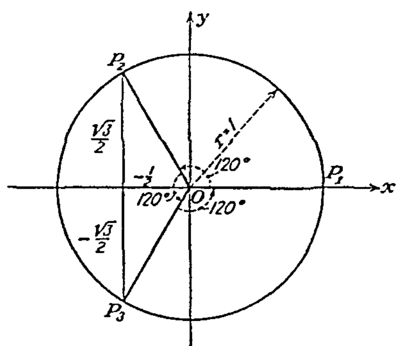


FIG. 1·7.

defines the angle φ/n, and by dividing the circumference into n equal parts starting from the point P_1. The points $P_1, P_2, P_3, \ldots, P_n$ thus obtained are the representative points of the n roots, since their coordinates are given by

$$\left.\begin{aligned} x &= \sqrt[n]{R}\cos\left(\frac{\varphi}{n} + k\frac{2\pi}{n}\right) \\ y &= \sqrt[n]{R}\sin\left(\frac{\varphi}{n} + k\frac{2\pi}{n}\right) \end{aligned}\right\} \qquad (k = 0, 1, 2, \ldots, n-1)$$

Figure 1·6 shows the graphic construction for the roots of the previous problem, while Fig. 1·7 shows the construction for the three cube roots of $Z = 1$:

$$Z = 1 + 0i = 1(\cos 0 + i \sin 0)$$

$$r = \sqrt[3]{1} = 1 \qquad \theta = \frac{0}{3} + k\frac{2\pi}{3} = 0°,\ 120°,\ 240° \quad (k = 0, 1, 2)$$

$$z_1 = 1 \qquad z_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \qquad z_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Noticing that $\sqrt[n]{a} = \sqrt[n]{a}\,\sqrt[n]{1}$, the nth roots of a *real* number are easily obtained by multiplying any *one* of its roots by the n nth roots of unity.

1·3 Variables and Functions

Numbers are used to define quantities whose value does not change. But very many quantities may or actually do change in value. For instance, time changes constantly. The temperature of a room may change. A quantity whose value may change is called a *variable.* It is customary to indicate variables by the last letters of the alphabet, x, y, z, t, w.

Very often a variable is found to have values that depend upon the values of another variable. For instance, the temperature of a room may change depending upon the time of the day. We say in this case that there is a *functional relationship* between the two variables, temperature T and time t. If we have a graph of the temperature versus time, we may select a value of the time t and locate on the graph the corresponding value of the temperature T. We say in this case that t is the *independent* variable, since it can be chosen freely, and T the *dependent* variable, since, once the value of t is chosen, the value of T depends upon this particular value of t and cannot be chosen freely. The dependent variable T is also said to be a *function* of t.

A function is therefore a variable depending on one or more other variables. The functional relationship thus established is symbolized by the equation

$$T = f(t) \tag{1·3·1}$$

The cost of a train ticket C depends upon the length of the traveled distance s. The functional relationship between C and s is symbolized by writing

$$C = f(s)$$

But since to one value of s there corresponds one value of C, and vice versa, we may consider s as the function and C as the variable and write

$$s = F(C)$$

The function F is said to be the *inverse* of the function f and is sometimes written as

$$s = f^{-1}(C)$$

If you want to travel a *given* distance, you are interested in $f(s)$. If you have a *given* amount of money that you wish to spend on a vacation trip, you are interested in $f^{-1}(C)$.

When a quantity w depends upon various variables $x,y,z, \ldots$, we write

$$w = f(x,y,z, \ldots)$$

to indicate the functional relationship between w and $x,y,z, \ldots$. In this course we shall consider only functions y of a single variable x and shall often indicate the functional relationship by the shorter symbol

$$y = y(x) \tag{1·3·2}$$

It will also be assumed, unless otherwise stated, that to each value of x in a given interval there corresponds one and only one value of y, that is, that y is a *single-valued* function of x.

It must be clearly understood that a functional relationship is *not* an *explanation* of why the function varies but purely a *description* of *how* it varies. Science does not explain, it only describes, phenomena.

A functional relationship may be represented in four ways.

1. By means of a verbal or written statement, as when we say: "The volume of a certain amount of gas is inversely proportional to its pressure."

2. By means of a table of corresponding values, as in a schedule of prices of railroad tickets.

3. By means of a graph, this being perhaps the presentation most commonly used in engineering.

4. By means of a mathematical formula, as when we write $y = 2x^2$.

The representation of a functional relationship by means of a formula is often the most fruitful, and all other representations may be derived from it; but when it is either difficult or impossible to locate the mathematical formula representing the function, other representations must be resorted to. In some cases, moreover, a function requires more than one mathematical formula for its representation: for instance, the function giving the relationship between postage and weight of first-class mail requires the following specification, where

$$p = \text{postage, in cents,} \qquad w = \text{weight, in ounces.}$$

$$\begin{array}{lll} p = 3 & \text{for} & 0 < w \leq 1 \\ p = 6 & \text{for} & 1 < w \leq 2 \\ p = 9 & \text{for} & 2 < w \leq 3 \\ \cdots & \cdots & \cdots \end{array}$$

This can also be written as

$$p = 3n \qquad \text{for} \qquad (n - 1) < w \leq n \qquad (n = 1, 2, \ldots)$$

The range of values of the variable in which we are interested forms the *interval of definition of the function.* For instance, the *real* function $y = \sqrt{4 - x^2}$ is defined in the interval $-2 \leq x \leq 2$, which is also represented by the symbol $(-2, 2)$ or by the "absolute-value" symbol $|x| \leq 2$. [*Read:* "*absolute value* of x less than or equal to 2"; see Eq. (1·1·3).]

Sometimes a variable z depends on another variable y, which in turn depends upon a third variable x. We write in this case

$$z = F(y) \qquad y = f(x)$$

or

$$z = F[f(x)]$$

or

$$z = z[y(x)] \tag{1·3·3}$$

and say that z is a *composite function* of x through the intermediate variable y. Thus the volume V of a gas depends upon the pressure P, but the pressure may vary because of a change in the temperature T; hence V is a composite function of T through P,

$$V = V[P(T)]$$

1·4 Limit of a Variable and Limit of a Function

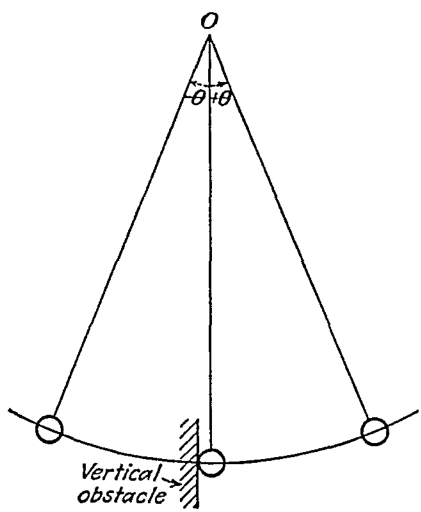


FIG. 1·8.

In a physics experiment a pendulum, consisting of a heavy mass attached to one end of a bar whose other end is hinged at a point O (Fig. 1·8), swings freely after being displaced 30 deg from its vertical position of equilibrium. Owing to friction at the hinge and to air resistance the amplitude of the oscillations decreases after each swing, as shown by the first line of Table 1·2, in which swings to the right are indicated by a plus sign and swings to the left by a minus sign.

Line 2 of the table records the successive swings of the same pendulum in a second experiment, during which the frictional resistances at the hinge are increased. In a third experiment a vertical obstacle is set to the left of O, so that the pendulum is compelled to bounce back after reaching the position $\theta = 0°$. Line 3 of the table shows that in this case the amplitude of the oscillations is always positive but decreases as in

the previous cases, because of frictional resistances and energy losses during impact against the obstacle.

In a fourth experiment, recorded on line 4 of Table 1·2, the pendulum swings in a thick fluid and the combined resistances vary in such a way that the pendulum stops in its vertical position of equilibrium after five swings.

TABLE 1·2

Pendulum	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	. . .
1	+30°	−20°	+13.33°	−8.89°	+5.93°	−3.95°	+2.63°	. . .
2	+30°	−15°	+ 7.5°	−3.75°	+1.88°	−0.94°	+0.47°	. . .
3	+30°	+15°	+ 7.5°	+3.75°	+1.88°	+0.95°	+0.47°	. . .
4	+30°	−20°	+12°	−6°	+2°	0°	0°	. . .

Table 1·2 shows four *different* ways in which the amplitude θ_i of the pendulum oscillations at the end of each swing approaches its final value, zero, but the tabulated values θ_i satisfy in *all* four cases the following two conditions:

1. They become smaller and smaller in absolute value.
2. They remain small.

In other words, the angles θ_i approach the value zero and *remain* near this value in all four cases.

When a variable x *approaches* (at least after a while) a value x_0 and *remains* near this value, we say that x *approaches* x_0 *as a limit* and write

$$\lim x = x_0$$

The pendulum swings θ_i in the four experiments approach zero as a limit.

Two negative circumstances must also be emphasized.

1. While in all four experiments the pendulum approaches a vertical position, this happens in a different way in each experiment. It is immaterial to the concept of limit of a variable *how* the limit is approached, provided that conditions (1) and (2) above be satisfied.

2. While in the fourth experiment the limit zero is actually reached, in the first three experiments an infinite number of swings is necessary to stop the pendulum. It is immaterial to the concept of limit of a variable *whether or not* the limit is reached. The connotation of impossibility or of extreme difficulty, inherent in the common usage of the word "limit," is entirely absent from its mathematical significance.

A constant x_0 can also be stated to be the limit of a variable x by either of the two mathematical notations

$$\lim |x - x_0| = 0$$
$$|x - x_0| < \delta$$

where δ is a number that can be made as small as we please.

Let us now perform a fifth experiment in which the resistances are so high that the pendulum cannot swing but approaches its vertical position of equilibrium as shown in Table 1·3, where the values of the angle θ are measured at time intervals of 1 sec.

TABLE 1·3

t	0	1	2	3	4	5	6	7	. . .
θ	+30°	+15°	+6°	+2°	+1°	0.4°	+0.2°	0°	. . .

If we consider the angle θ as time increases, θ becomes a well-defined function of t, which *approaches* and *remains* near the value zero as the variable t approaches the value 7 *as a limit*. We say in this case that the function θ approaches zero *as a limit* as t approaches 7 and write

$$\lim_{t \to 7} \theta = 0$$

In general, if a function $y(x)$ approaches a value y_0 as a limit when x approaches x_0, we write

$$\lim_{x \to x_0} y = y_0$$

This result can also be stated by saying that, if x is taken near enough to x_0, y can be made as near to y_0 as we wish. Mathematically this means that $|y - y_0| < \epsilon$ (ϵ as small as we please), provided $|x - x_0| < \delta$ (δ a suitably small number).

Here again it is immaterial whether the function does or does not *reach* the limit. The concept of limit of a function is, so to speak, a kinematical concept, which investigates *the behavior of the function as the variable approaches its limit* and not the value of the function *at* the limit. It may be noticed that the angle θ of the first experiment, *considered as a function of the continuous variable* t, does not approach zero as a limit, since it becomes zero and then increases again, *i.e.*, does not *remain* near zero.

The only essential difference between the limit of a variable and the limit of a function is that the law of approach of a function cannot be chosen freely, since it depends upon the law of variation of the independent variable.

If a function increases indefinitely as the variable approaches a given value, we say that the limit of the function is infinity. For example, $y = 1/(x - 2)^2$ approaches infinity as $x \to 2$ (Fig. 1·12a). Some functions do not approach any limit for certain values of the variable. Thus

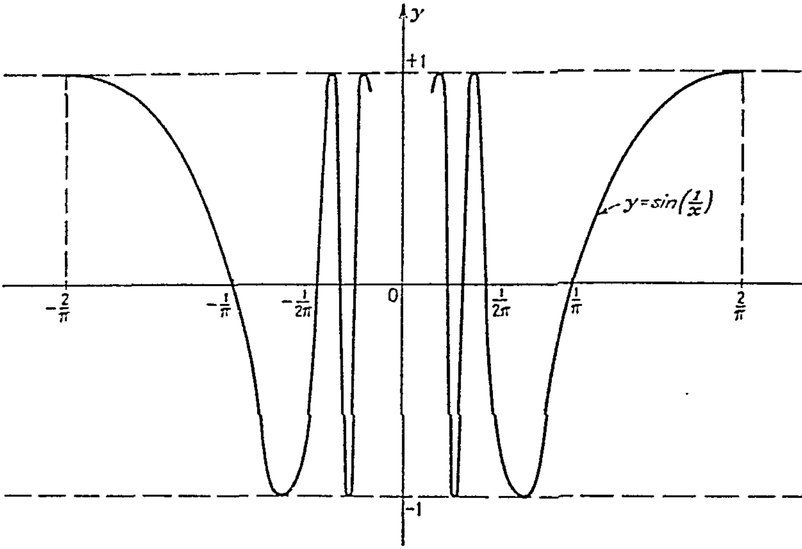


FIG. 1·9.

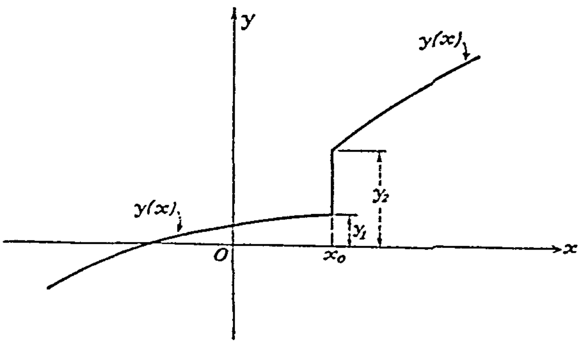


FIG. 1·10.

$y = \sin(1/x)$ keeps oscillating between 1 and -1 an infinite number of times as $x \to 0$ (Fig. 1·9). Other functions have different limits from the left and from the right, as shown in the graph of Fig. 1·10. In this case we write

$$\lim_{x \to x_0^-} y = y_1$$
$$\lim_{x \to x_0^+} y = y_2$$

1·5 Continuity

The graph of Fig. 1·11 gives the heights h (in feet) above sea level of the bottom of a creek versus horizontal distances x, measured along the creek from a certain origin. The creek has a waterfall at $x = 4$ miles. We wish to locate a stretch of creek in the neighborhood of $x = 2$ miles for which the altitude h will be less than 90 ft and more than 80 ft. It is clear from the graph that, provided that $1 < x < 3$, then $80 < h < 90$; or, in mathematical symbols, that

$$|h - 85| < 5 \qquad \text{when} \qquad |x - 2| < 1$$

and that the difference in level can be made smaller by limiting the values

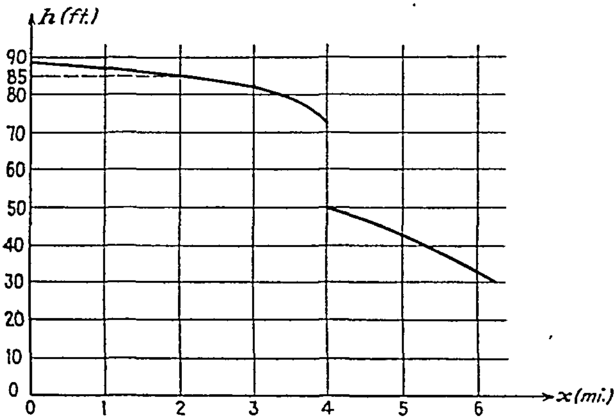


FIG. 1·11.

of x to a smaller interval around $x = 2$. Whenever the variation of a function can be limited by limiting the variation of the variable around a given point, the function is said to be *continuous at that point.* Our function $h(x)$ is continuous at $x = 2$.

It will also be noticed that, as x approaches 2, the function h approaches 85, which is the value of h *at* $x = 2$,

$$\lim_{x \to 2} h(x) = h(2) = 85$$

Hence another way of stating mathematically that a function $y(x)$ is continuous at $x = x_0$ is to write that

$$\lim_{x \to x_0} y = y(x_0)$$

i.e., that the limit of the function equals the value of the function *at* the limit. An equivalent mathematical statement of continuity is obtained by transposing $y(x_0)$ to the left-hand member of the equation

and by writing

$$\lim_{x \to x_0} |y(x) - y(x_0)| = 0$$

If we now try to limit the altitudes of the creek in the neighborhood of $x = 4$ miles, we see that the variation of h cannot be made less than 20 ft, since even two points immediately to the right and left of $x = 4$ have a difference in level greater than 20 ft. Whenever the variation of a function cannot be limited by limiting the variation of the variable

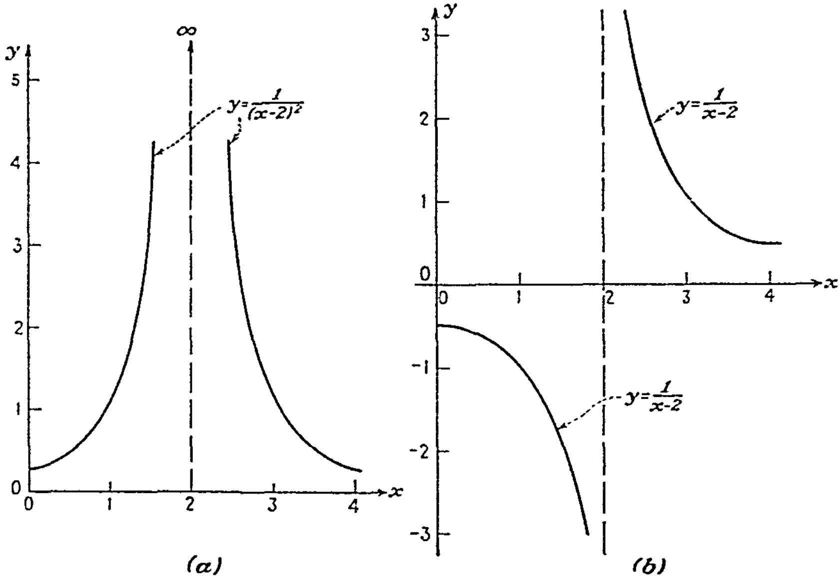


FIG. 1·12.

around a given point, the function is said to be *discontinuous* at that point. At points of discontinuity, the function jumps suddenly from one value to another, and there are actually two limits of the function, one from the left and one from the right (see Sec. 1·4).

A function $y(x)$ is also said to be discontinuous at a point x_0 if it approaches plus or minus infinity as x approaches x_0. For example, the function $y = 1/(x - 2)^2$ approaches infinity as x approaches 2 (Fig. 1·12a). The function $y = 1/(x - 2)$ approaches plus infinity as x approaches 2 from the right and minus infinity as x approaches 2 from the left (Fig. 1·12b).

1·6 Infinitesimals

Many formulas used in engineering computations are approximate formulas valid only for a limited range of values of the variable. For

instance, it can be proved (see Sec. 5·8*d*) that, for small values of x,

$$\sqrt{1+x} \doteq 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 \tag{a}$$

where [$\doteq$] stands for "approximately equal to." In Table 1·4 the last two terms of Eq. (*a*) and the approximate and precise values of $\sqrt{1+x}$ are tabulated for decreasing values of x. The table proves that formula

TABLE 1·4

x	$\frac{x}{2}$	$\frac{x^2}{8}$	$1+\frac{x}{2}-\frac{x^2}{8}$	$\sqrt{1+x}$
1.0	0.5	0.125	1.375	1.4142
0.5	0.25	0.03125	1.21875	1.2247
0.1	0.05	0.00125	1.04875	1.0488
0.01	0.005	0.0000125	1.0049875	1.0050
0.001	0.0005	0.000000125	1.0004999	1.0005

(*a*) gives five correct significant figures for x less than 0.1 and that for x less than 0.01 the contribution of the term $x^2/8$ can be neglected in comparison with the contribution of $x/2$, if no more than five correct figures are needed, because, as x approaches zero, the term $x^2/8$ becomes small more rapidly than the term $x/2$.

This vague statement can be put in a more definite mathematical form by means of the concept of infinitesimal. *An infinitesimal is a function of x that approaches zero as a limit as x approaches a given value x_0.* For instance,

$$y = \log (x - 3)$$

is an infinitesimal as x approaches 4, since $\log (4 - 3) = \log 1 = 0$. The functions $x/2$ and $x^2/8$ are both infinitesimals as x approaches zero. An infinitesimal is *not* a small quantity but a quantity that *becomes indefinitely smaller.*

We noticed that, while both $x/2$ and $x^2/8$ approach zero as x approaches zero, $x^2/8$ approaches zero faster than $x/2$. Knowledge of the "speed" of an infinitesimal is of fundamental importance in engineering theory and in engineering computations because it allows simplification of formulas and reduction of computations. For example, formula (*a*) can be written as

$$\sqrt{1+x} = 1 + \tfrac{1}{2}x$$

as soon as we know that the term $x^2/8$ becomes negligible. The relative speed, or *order*, of two functions $f(x)$ and $g(x)$, both infinitesimals as

x approaches x_0, is measured by taking the limit of their ratio as x approaches x_0. There are three possible values of this limit.

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \begin{cases} 0 \\ k \text{ (a finite number } \neq 0) \\ \infty \end{cases}$$

In the first case, $f(x)$ is faster than $g(x)$, that is, the order of $f(x)$ is *higher* than the order of $g(x)$; in the second, $f(x)$ and $g(x)$ have the same speed (*i.e.*, they are infinitesimals of the *same* order); in the third case, $g(x)$ is faster than $f(x)$, that is, the order of $f(x)$ is *lower* than the order of $g(x)$. For example, if $f(x) = x^2/8$ and $g(x) = x/2$,

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x^2/8}{x/2} = \lim_{x \to 0} \frac{x}{4} = 0$$

and $x^2/8$ is of higher order than $x/2$. If $f(x) = 24x^2$ and $g(x) = 12x^2$,

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{24x^2}{12x^2} = \lim_{x \to 0} \frac{24}{12} = 2$$

and $24x^2$ is of the same order as $12x^2$. If $f(x) = 3x$ and $g(x) = 2x^2$,

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{3x}{2x^2} = \lim_{x \to 0} \frac{3}{2x} = \infty$$

and the order of $3x$ is less than the order of $2x^2$.

This qualitative correspondence of infinitesimals' speeds can be made quantitative by choosing a series of fundamental infinitesimals with increasing speeds and by comparing the speed of the given infinitesimal with these speeds. The function $y = x - x_0$ and its powers, which are infinitesimals as x approaches x_0, are usually chosen as comparative infinitesimals. *An infinitesimal of the same order as $(x - x_0)^n$ is called an infinitesimal of the nth order.* For example, $y = \sqrt{x - 1}$ is an infinitesimal of order $\frac{1}{2}$ as x approaches 1.

The sum of two infinitesimals is obviously an infinitesimal; and if one of the two is of higher order, it can be dropped as soon as x is near its limit. Thus $(x/2) - (x^2/8)$ is an infinitesimal as x approaches zero, and the infinitesimal of the second order $x^2/8$ can be dropped in comparison with the first-order infinitesimal $x/2$ whenever x is sufficiently small. How soon it may be dropped depends upon the accuracy required by the particular problem at hand.

The lowest-order part of an infinitesimal is called its *principal part:* $x/2$ is the principal part of $(x/2) - (x^2/8)$. In comparing infinitesimals only their principal parts need be compared. For example, given,

$f(x) = 2x + 3x^3$ and $g(x) = \sqrt{3x + 2x^2}$,

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{2x + 3x^3}{\sqrt{3x + 2x^2}} = \lim_{x\to 0} \frac{2x}{\sqrt{3x}} = \lim_{x\to 0} \frac{2}{\sqrt{3}} \sqrt{x} = 0$$

The order of $f(x)$ is higher than the order of $g(x)$; in fact, the order of $f(x)$ is 1, while the order of $g(x)$ is $\frac{1}{2}$.

1·7 Derivatives

a. Definition

If I have some doubts about the accuracy of the speedometer of my car, I may use the following test, which has been devised to check it: I drive with two friends along a highway and drop one at milestone 15 and the other at milestone 16, giving them two synchronized watches. I then drive a little farther and come back at a speed, as constant as possible, that my speedometer shows to be 40 mph. My second friend clocks my passage at 4:02, my first at 4:04. Since it took 2 min ($\frac{2}{60}$ hr) to drive 1 mile, my average speed was actually

$$v_{av} = \frac{1}{\frac{2}{60}} = 30 \text{ mph}$$

Inasmuch as this check is not entirely accurate, for a perfectly constant speed of 40 mph is hard to maintain for two full minutes, I station my second friend $\frac{1}{2}$ mile from milestone 15 and repeat the test. The times are now 4:15 and 3 sec, and 4:15 and 53 sec, giving a speed in the second trial of

$$v_{av} = \frac{0.5}{50} \times 3600 = 36 \text{ mph}$$

which checks much better the speedometer reading. The accuracy of the test can be further improved if the two friends are set nearer and nearer so as to make the driving time shorter and shorter, but a perfect check can be obtained only by measuring the instantaneous value of the speed.

All this can be rigorously and clearly written in mathematical terms. Let $s(t)$ be the distance traveled by the car in a time t, s and t being measured from the beginning of motion, and let t_1 and t_2 be the values of the time when the car goes by the first and the second friend. The checked distance being $s(t_2) - s(t_1)$ and the time taken to cover it being $t_2 - t_1$, the average speed is

$$v_{av} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

By means of the new symbol for the *difference*, or *increment*, of the vari-

able t,

$$\Delta t = t_2 - t_1$$

(*read:* "delta t equals $t_2 - t_1$"), $t_2 = t_1 + \Delta t$, and v_{av} may be written as

$$v_{av} = \frac{s(t_1 + \Delta t) - s(t_1)}{\Delta t}$$

In order to get an ideal check, the time Δt must become smaller and smaller, *i.e.*, be an infinitesimal. Hence the instantaneous value of the speed v is obtained by taking the limit of v_{av} as $\Delta t \to 0$,

$$v(t_1) = \lim_{\Delta t \to 0} \frac{s(t_1 + \Delta t) - s(t_1)}{\Delta t}$$

When this limit exists, the function $s(t)$ is said to be *differentiable at* $t = t_1$ and the limit v is called the *derivative* of $s(t)$ at $t = t_1$ and is indicated by one of the following symbols:

$$\left.\frac{ds}{dt}\right]_{t=t_1} \qquad s_1' \qquad \dot{s}_1$$

The derivative of a function $y = f(x)$ at $x = x_1$ is then, *by definition*,

$$\left.\frac{dy}{dx}\right]_{x=x_1} = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \tag{1·7·1}$$

dy/dx, which must be considered as a single symbol and *not as a fraction*, is a shorthand expression for the sequence of operations indicated at the right-hand member of Eq. (1·7·1),

1. Subtract the value of $f(x)$ at $x = x_1$ from the value of $f(x)$ at $x = x_1 + \Delta x$.
2. Divide by Δx.
3. Take the limit as Δx approaches zero.

Hence the definition of the derivative, Eq. (1·7·1), gives also a method for its computation, sometimes called the "Δ method."

The difference $f(x + \Delta x) - f(x)$ is the change in the function $y = f(x)$ due to a change Δx in the variable x and is often indicated by Δf or Δy. The ratio $\Delta y/\Delta x$ therefore measures the *average rate of change* of the function with respect to the variable, and dy/dx measures *the instantaneous rate of change* of y with respect to x. The speed v, that is, the instantaneous rate of change of the mileage with respect to the time, is measured by the derivative of s with respect to t.

The importance of the concept of limit in the definition of derivative can now be fully appreciated. If we *make* $\Delta x = 0$ in Eq. (1·7·1), the

ratio

$$\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

takes the form 0/0, which has no meaning; but if we take the limit as Δx approaches zero, we actually get the *behavior of this ratio as Δx approaches zero.*

The concept of derivative was systematically used and investigated by Newton and Leibnitz independently toward the end of the seventeenth century. It is one of the two basic concepts of the calculus and allows the study of the important field of problems in which instantaneous rates must be dealt with.

b. Computation of Derivatives by the Δ Method

We shall now compute by the Δ method the derivatives of three elementary functions:

(1)
$$\begin{aligned}
y(x) &= 2 + 3x - x^2 \\
y(x + \Delta x) &= 2 + 3(x + \Delta x) - (x + \Delta x)^2 \\
&= 2 + 3x + 3\Delta x - [x^2 + 2x\,\Delta x + (\Delta x)^2] \\
\Delta y &= y(x + \Delta x) - y(x) = 3\Delta x - 2x\,\Delta x - (\Delta x)^2 \\
\frac{\Delta y}{\Delta x} &= 3 - 2x - \Delta x \\
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 3 - 2x
\end{aligned}$$

(2)
$$\begin{aligned}
y(x) &= \log_b x \\
y(x + \Delta x) &= \log_b (x + \Delta x) \\
\Delta y &= \log_b (x + \Delta x) - \log_b x = \log_b \frac{x + \Delta x}{x} = \log_b \left(1 + \frac{\Delta x}{x}\right) \\
\frac{\Delta y}{\Delta x} &= \frac{1}{\Delta x} \log_b \left(1 + \frac{\Delta x}{x}\right)
\end{aligned}$$

or, letting $t = x/\Delta x$, and hence $1/\Delta x = t/x$,

$$\frac{\Delta y}{\Delta x} = \frac{t}{x} \log_b \left(1 + \frac{1}{t}\right) = \frac{1}{x} \log_b \left(1 + \frac{1}{t}\right)^t$$

Noticing that, as $\Delta x \to 0$, $t \to \infty$, since x is kept constant in this limiting process,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{t \to \infty} \frac{1}{x} \log_b \left(1 + \frac{1}{t}\right)^t = \frac{1}{x} \log_b \left[\lim_{t \to \infty} \left(1 + \frac{1}{t}\right)^t \right]$$

The limit in the square bracket can be proved to be an irrational number, usually indicated by the letter e ($e = 2.718\ .\ .\ .$), by means of

which

$$\frac{dy}{dx} = \frac{1}{x} \log_b e$$

Many formulas of the calculus would be cumbersome if the multiplying numerical constant $\log_b e$ should appear in the expression for the derivative of the logarithm. To eliminate this constant, the base b of logarithms in calculus is always taken equal to e, $\log_b e$ becomes $\log_e e = 1$, and the derivative of the logarithmic function simplifies to

$$\frac{d \ln x}{dx} = \frac{d}{dx} \log_e x = \frac{1}{x}$$

The $\log_e x$ is commonly indicated by the symbol $\ln x$ (natural logarithm of x).

(3)
$$\begin{aligned} y(x) &= \sin x \\ y(x + \Delta x) &= \sin (x + \Delta x) \\ \Delta y &= y(x + \Delta x) - y(x) = \sin (x + \Delta x) - \sin x \end{aligned}$$

Using the trigonometric identity

$$\sin \beta - \sin \alpha = 2 \sin \frac{\beta - \alpha}{2} \cos \frac{\beta + \alpha}{2}$$

with $\beta = x + \Delta x$ and $\alpha = x$, Δy becomes

$$\Delta y = 2 \sin \frac{\Delta x}{2} \cos \left(x + \frac{\Delta x}{2}\right)$$

and hence

$$\frac{\Delta y}{\Delta x} = \frac{\sin (\Delta x/2)}{\Delta x/2} \cos \left(x + \frac{\Delta x}{2}\right)$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin (\Delta x/2)}{\Delta x/2} \cos \left(x + \frac{\Delta x}{2}\right)$$

and, since

$$\lim_{\Delta x \to 0} \cos \left(x + \frac{\Delta x}{2}\right) = \cos x$$

$$\lim_{\Delta x \to 0} \frac{\sin (\Delta x/2)}{\Delta x/2} = 1$$

provided that $\Delta x/2$ be *measured in radians* (see Sec. 1·8 *d*2),[1]

$$\frac{dy}{dx} = \cos x$$

[1] The radian is always used as the unit of angles in the calculus in order that the derivatives of the trigonometric functions be free of a numerical multiplying constant deriving from the limit considered above.

Table 1·5 gives the derivatives of the elementary functions most often used in engineering computations.

TABLE 1·5

y	$\dfrac{dy}{dx}$
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\dfrac{1}{\cos^2 x}$
e^x	e^x
$\ln x$	$\dfrac{1}{x}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\dfrac{1}{\cosh^2 x}$

c. Differentiation Rules

The reader should review the fundamental rules of differentiation given in elementary books on calculus and should practice using them in various problems. Three of these rules, needed in almost all differentiation problems, will be derived here.

1. *Derivative of the Product of Two Functions.*

$$\begin{aligned}
y(x) &= u(x)v(x) \\
u(x + \Delta x) - u(x) &= \Delta u \\
v(x + \Delta x) - v(x) &= \Delta v \\
y(x + \Delta x) &= u(x + \Delta x) \cdot v(x + \Delta x) \\
&= (u + \Delta u)(v + \Delta v) \\
&= uv + u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v \\
\Delta y &= u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v \\
\frac{\Delta y}{\Delta x} &= u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x} \\
\frac{dy}{dx} &= u\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x} + v\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} + \lim_{\Delta x\to 0}\Delta u\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}
\end{aligned}$$

But $\Delta u \to 0$ as $\Delta x \to 0$, and hence

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \tag{1·7·2}$$

Examples

1. $y = x^2 \sin x \qquad u = x^2 \qquad v = \sin x$
   $y' = x^2 \cos x + 2x \sin x$
2. $y = \sqrt{x}\log x \qquad u = \sqrt{x} \qquad v = \log x$
   $y' = \sqrt{x}\dfrac{1}{x} + \dfrac{1}{2}\dfrac{1}{\sqrt{x}}\log x = \dfrac{1}{\sqrt{x}}\left(1 + \frac{1}{2}\log x\right)$

2. *Derivative of the Function of a Function.* Given $z = F(y)$, where $y = f(x)$, when x increases by Δx, y increases by Δy, and hence z increases by Δz.

$$y(x + \Delta x) = y(x) + \Delta y$$
$$z(y + \Delta y) = z(y) + \Delta z$$

$$\frac{dz}{dx} = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta y}\frac{\Delta y}{\Delta x} \qquad \text{(multiplying and dividing by } \Delta y)$$

But $\Delta y \to 0$ as $\Delta x \to 0$; hence

$$\frac{dz}{dx} = \lim_{\Delta y \to 0} \frac{\Delta z}{\Delta y} \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dz}{dy} \cdot \frac{dy}{dx} \tag{1·7·3}$$

Examples

1. $z = \sin 4x^2 \qquad y = 4x^2 \qquad z = \sin y$

   $\dfrac{dz}{dx} = \cos y \cdot 8x = 8x \cos 4x^2$

2. $z = \sqrt{\sin x} \qquad y = \sin x \qquad z = \sqrt{y}$

   $\dfrac{dz}{dx} = \dfrac{1}{2}\dfrac{1}{\sqrt{y}} \cos x = \dfrac{1}{2}\dfrac{\cos x}{\sqrt{\sin x}}$

3. $z = (x + \cos x^2)^3 \qquad y = x + \cos x^2 \qquad z = y^3$

   $\dfrac{dz}{dy} = 3y^2 \qquad \dfrac{dy}{dx} = 1 - 2x \sin x^2$

   $\dfrac{dz}{dx} = 3(x + \cos x^2)^2(1 - 2x \sin x^2)$

3. *Derivative of an Inverse Function.* Given $y = f(x)$, we call $x = g(y)$, the inverse of $f(x)$. Noticing that $\Delta y \to 0$ as $\Delta x \to 0$, we can write

$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x/\Delta y} = \frac{1}{\lim\limits_{\Delta y \to 0} \Delta x/\Delta y}$$
$$= \frac{1}{dx/dy} = \frac{1}{g'(y)} \tag{1·7·4}$$

In words, the derivative of the inverse function is equal to the reciprocal of the derivative of the function.

Examples

1. $y = \ln x \qquad x = e^y$

   $\dfrac{dy}{dx} = \dfrac{1}{x} \qquad \dfrac{dx}{dy} = x = e^y$

2. $y = \sin x \qquad x = \arcsin y$

   $\dfrac{dy}{dx} = \cos x \qquad \dfrac{dx}{dy} = \dfrac{1}{\cos x} = \dfrac{1}{\sqrt{1 - y^2}}$

By means of this rule the derivatives of the inverse trigonometric and inverse hyperbolic functions,[1] given in Table 1·6, are easily obtained.

[1] The inverse hyperbolic functions $\sinh^{-1} x$, $\cosh^{-1} x$, and $\tanh^{-1} x$ are also represented by the symbols argsinh x, argcosh x, and argtanh x. (*Read:* "argument sinh x. etc.")

TABLE 1·6

y	$\dfrac{dy}{dx}$
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\arccos x$	$\dfrac{-1}{\sqrt{1-x^2}}$
$\arctan x$	$\dfrac{1}{1+x^2}$
$\operatorname{argsinh} x$	$\dfrac{1}{\sqrt{x^2+1}}$
$\operatorname{argcosh} x$	$\dfrac{1}{\sqrt{x^2-1}}$
$\operatorname{argtanh} x$	$\dfrac{1}{1-x^2}$

d. Higher Derivatives and Partial Derivatives

The derivative of a function $f(x)$, as a rule, is another function of x, which, in general, can be differentiated. For example, if $y = \sin x$, $y' = \cos x$ and the function $\cos x$ is differentiable. Its derivative, $-\sin x$, is called the *second derivative* of $\sin x$. The derivative of the second derivative of y is called the *third derivative* of y, and the function obtained by n successive differentiations is called the *nth derivative* of y. The following symbols are used to indicate the successive derivatives of y:

$$\begin{array}{ccccc} \dfrac{dy}{dx} & \dfrac{d^2y}{dx^2} & \dfrac{d^3y}{dx^3} & \cdots & \dfrac{d^ny}{dx^n} \\ y' & y'' & y''' & \cdots & y^{(n)} \end{array}$$

Given a function z of two variables x,y, the derivative computed by keeping y constant and varying only x is called the *partial derivative of z with respect to x* and is symbolized by $\dfrac{\partial z}{\partial x}$. Similarly, $\dfrac{\partial z}{\partial y}$ is the derivative of z computed by keeping x constant and varying only y. For example, given $z = x^2 + y^2 + xy$,

$$\frac{\partial z}{\partial x} = 2x + y \qquad \frac{\partial z}{\partial y} = 2y + x$$

The partial derivative $\dfrac{\partial z}{\partial x}$ is, in general, a function of x and y and can usually be differentiated again partially with respect to x to give the *second partial derivative of z with respect to x*, $\dfrac{\partial^2 z}{\partial x^2}$; but it may also be differentiated partially with respect to y to give the *second mixed derivative* $\dfrac{\partial^2 z}{\partial y\, \partial x}$.

In the previous example,

$$\frac{\partial^2 z}{\partial x^2} = 2 \qquad \frac{\partial^2 z}{\partial y^2} = 2 \qquad \frac{\partial^2 z}{\partial y\,\partial x} = 1$$

Differentiating $\frac{\partial z}{\partial y}$ partially with respect to x, we get

$$\frac{\partial^2 z}{\partial x\,\partial y} = 1$$

It is true of most functions met in engineering problems that

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$$

As previously mentioned, we shall restrict our treatment almost exclusively to functions of *one* variable and hence to ordinary derivatives.

1·8 Differentials

a. Geometrical Interpretation of the Derivative

The graph of Fig. 1·13 gives the elevations y of a stretch of road versus horizontal distances along the road from a given origin O. We

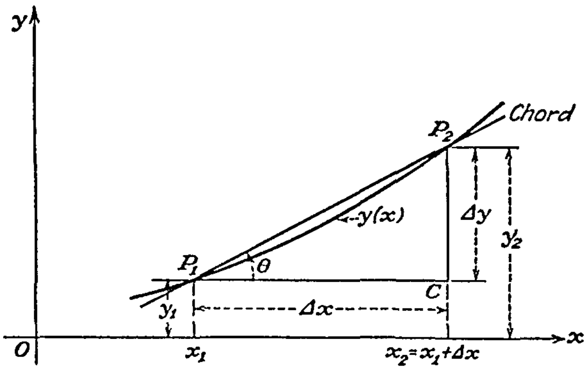


FIG. 1·13.

wish to compute from this graph the slope of the road at a point P_1, to check whether the road can be managed by a truck. Taking another point P_2 on the road, a horizontal distance Δx to the right of P_1, and calling Δy the difference in level between P_2 and P_1, the *average* slope of the road between P_1 and P_2 is

$$m_a = \frac{CP_2}{P_1C} = \frac{\Delta y}{\Delta x} = \tan\theta$$

where θ is the angle between the chord P_1P_2 and the positive x axis. In order to get the slope *at* P_1, the point P_2 is made to slide along the road

toward P_1, that is, Δx is made an infinitesimal. As $\Delta x \to 0$, the chord P_1P_2 approaches a limiting position, which is the position of the tangent to the road at P_1; hence the slope of this tangent, which is by definition the slope of the road, is given by

$$m_1 = \lim_{\Delta x \to 0} m_a = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \left.\frac{dy}{dx}\right]_{x=x_1} = \tan \varphi \tag{1·8·1}$$

where φ is the angle between the tangent to the road and the x axis (Fig. 1·14). In words, *the slope of the graph of a function $y(x)$ at a point x_1 is measured by the value of the derivative of y at x_1.* For instance, the slope of the graph of $y = x^2$ at $x = 0.5$ equals

$$\left.\frac{d(x^2)}{dx}\right]_{x=0.5} = 2x\Big]_{x=0.5} = 1$$

as may also easily be checked graphically.

b. The Differential

The tangent to a curve $y(x)$ at a point $P_1(x_1,y_1)$ (see Fig. 1·14) intersects the vertical through P_2 at a point D, whose distance from the point

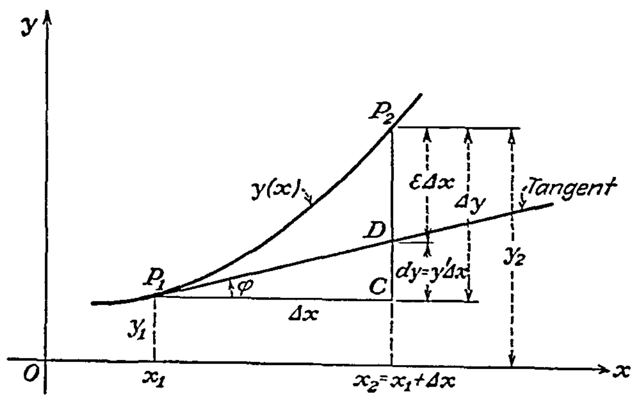


FIG. 1·14.

C equals

$$CD = P_1C \tan \varphi = \tan \varphi \cdot \Delta x$$

or, by Eq. (1·8·1), using the symbol y' to indicate the derivative of y,

$$CD = y' \, \Delta x$$

The segment DP_2 has a length

$$DP_2 = CP_2 - CD = \Delta y - y' \, \Delta x = \left(\frac{\Delta y}{\Delta x} - y'\right) \Delta x \tag{a}$$

But the definition of derivative [Eq. (1·7·1)]

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

which may also be written as

$$\lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} - y' \right) = 0$$

shows that the quantity

$$\epsilon = \frac{\Delta y}{\Delta x} - y' \tag{b}$$

is an infinitesimal as Δx approaches zero. By Eq. (*b*), Eq. (*a*) becomes

$$DP_2 = \epsilon \, \Delta x$$

and proves that the distance DP_2, being the product of two infinitesimals as Δx approaches zero (ϵ and Δx itself), is an infinitesimal of higher order with respect to Δx (see Sec. 1·6).

Therefore by Eq. (*a*), as Δx becomes smaller and smaller, $y' \, \Delta x$ becomes very rapidly equal to Δy and can be taken as an *approximate* measure of Δy.

The quantity $y' \, \Delta x$ is called the *differential* of y and is symbolized by dy. The differential of a function y is therefore by definition the *product of the derivative of y times the increment of the variable x,*

$$dy = y' \, \Delta x \tag{1·8·2}$$

Since dy is very often easier to evaluate than Δy, it is substituted for Δy in engineering computations whenever Δx is very small. For example, if $y = x^2$, the difference between Δy and dy becomes

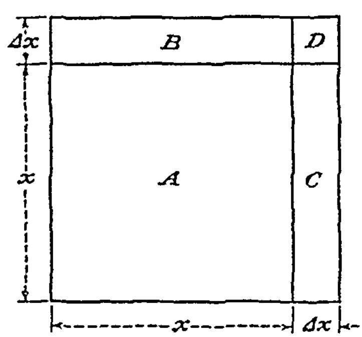


FIG. 1·15.

$$\Delta y = (x + \Delta x)^2 - x^2 = 2x \, \Delta x + (\Delta x)^2$$
$$dy = y' \, \Delta x = 2x \, \Delta x$$
$$\Delta y - dy = (\Delta x)^2$$

$(\Delta x)^2$ is an infinitesimal of the second order with respect to Δx and can be neglected as soon as Δx is small. This is geometrically demonstrated in Fig. 1·15, where the area y of the square A (x on the side) equals x^2, the area of the larger square [($x + \Delta x$) on the side] is $(x + \Delta x)^2$, and hence Δy is the sum of the areas of the two equal rectangles B and C and of the square D. As soon as Δx becomes small in comparison with x, the area of D becomes negligible in comparison with the areas of B and C; for instance, for $x = 10$,

$$\Delta x = 1, \qquad \Delta y = (101)^2 - (100)^2 = 201 \qquad dy = 2 \times 100 \times 1 = 200$$
$$\Delta y - dy = 1 \ll 201 = \Delta y$$

c. The Leibnitzian Symbol for the Derivative

The symbol dy is an "operational" shorthand symbol indicating the following operations: (1) take the derivative of y; (2) multiply by Δx.

Since differentials are used almost exclusively when Δx is an infinitesimal, it is customary to substitute the symbol dx for Δx in order to indicate that Δx approaches zero.[1] Writing Eq. (1·8·2) as

$$dy = y'\,dx$$

and dividing both sides by dx, we find that the symbol for the derivative

$$y' = \frac{dy}{dx}$$

can now be interpreted as *the ratio of the differential of the function to the differential of the variable.*

The symbol dy/dx, first used by Leibnitz, which (as pointed out in Sec. 1·7) had to be then considered as a single entity, can now be considered instead as a ratio of differentials. This result greatly simplifies many derivations of the calculus.

The differential of the differential of y is called the *second differential* of y and is symbolized by d^2y. To compute its value we perform twice successively the operations indicated by the d operator,

$$d^2y = d(dy) = d(y'\,dx) = \frac{d(y'\,dx)}{dx}\,dx = y''(dx)^2$$

where y'' indicates the second derivative of y.

Dividing by $(dx)^2$ both sides,

$$y'' = \frac{d^2y}{(dx)^2}$$

It is customary to drop the parentheses in the denominator of this equation and to write

$$y'' = \frac{d^2y}{dx^2}$$

It is thus seen that the second derivative of y can be considered as the ratio of the *second* differential of y to the *square* of the differential of x.

Similarly, the nth derivative of y can be considered as the ratio of

[1] This is permissible because the definition (1·8·2) of dy, which holds for any function y, gives in particular, for $y = x$,

$$dy = dx = y'\,\Delta x = 1 \cdot \Delta x = \Delta x$$

and shows that the differential dx of the independent variable is identical with its increment Δx.

the *nth differential* of y to the *nth power* of dx,

$$y^{(n)} = \frac{d^n y}{dx^n}$$

d. Applications of the Differential

1. One pound of air is contained in a cylinder of volume V at a pressure $P = 1000$ lb per sq in. If the pressure is slowly increased by 2 lb per sq in., by how much will the volume of the gas decrease?

It is well known from physics that the product of the volume and the pressure for an ideal gas is a function of the temperature T,

$$PV = f(T)$$

If the change in pressure occurs very slowly, the temperature of the gas does not change and the product PV equals a constant, say K,

$$PV = K$$

Hence we can write the volume as a function of the pressure,

$$V = \frac{K}{P}$$

A change of 2 lb per sq in. can be considered as practically small in comparison with the initial pressure of 1000 lb per sq in. Hence the corresponding change in V can be evaluated by means of the differential of V rather than by its increment,

$$\Delta V \doteq dV = \frac{dV}{dP}\,dP = -\frac{K}{P^2}\,dP$$

For 1 lb of air at 27°F, $K = 312{,}000$ lb in., and the change in volume becomes

$$\Delta V \doteq -\frac{312{,}000 \times 2}{1000^2} = -0.624 \text{ cu in.}$$

(Air may be considered as an ideal gas in an approximate computation.)

2. *Indeterminate Forms.* A corporation owns two grocery stores, which are opened on the same day. The two stores lose money at first, break even on the fifteenth day, and then proceed to make money. The company statistician finds that the incomes of the two stores follow the laws

$$y_1 = \frac{157}{15}(t - 15)$$

$$y_2 = 100 \sin \frac{\pi}{60}(t - 15)$$

where t is the time measured in days from the opening day (see Fig. 1·16). The owners wish to know the ratio R of the gains (or losses) of the two stores on each day. This can be easily computed for the first 14 days, but on the fifteenth day this ratio becomes

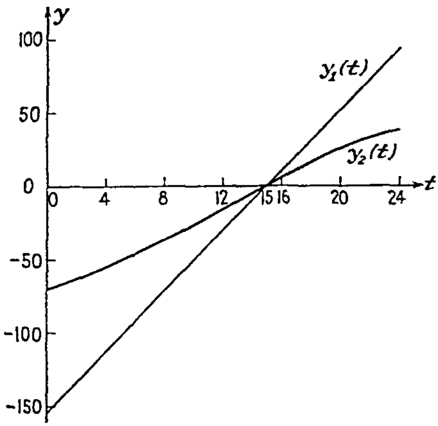


FIG. 1·16.

$$R_{15} = \frac{y_2(15)}{y_1(15)} = \frac{157(15-15)/15}{100 \sin(\pi/60)(15-15)} = \frac{0}{0}$$

an expression that has no meaning, since division by zero is not defined in algebra.

What the company statistician can do in this case is to compute the limit of R *as t approaches* 15, rather than to compute the value of R *at* $t = 15$.

$$\lim_{t\to 15} R = \lim_{\Delta t\to 0} \frac{y_2(15+\Delta t)}{y_1(15+\Delta t)} = \lim_{\Delta t\to 0} \frac{y_2(15)+\Delta y_2}{y_1(15)+\Delta y_1} = \lim_{\Delta t\to 0} \frac{\Delta y_2}{\Delta y_1}$$

since $y_2(15) = y_1(15) = 0$.

But as $\Delta t \to 0$, Δy approaches $dy = y'\,dt$; hence

$$\lim_{t\to 15} R = \lim_{\Delta t\to 0} \frac{\Delta y_2}{\Delta y_1} = \frac{dy_2}{dy_1} = \frac{y_2'(15)\,dt}{y_1'(15)\,dt} = \left.\frac{y_2'}{y_1'}\right]_{t=15}$$

or

$$\lim_{t\to 15} \frac{y_2}{y_1} = \left.\frac{y_2'}{y_1'}\right]_{t=15}$$

The ratio of two functions, both of which become zero for a given value of the variable, equals the ratio of their derivatives, computed for the same value of the variable. (Notice that the ratio of the derivatives and *not* the derivative of the ratio must be taken.) This result is called *de l'Hospital's rule* and is used in determining the value of ratios of the type 0/0 (*indeterminate forms*).

Applying de l'Hospital's rule to our problem, we find

$$\lim_{t\to 15} R = \left.\frac{157/15}{100/60\,\pi \cos(\pi/60)(t-15)}\right]_{t=15} = 2$$

In some cases the ratio y_2'/y_1' is also of the type 0/0. De l'Hospital's rule can then be applied to this new ratio and, if necessary, to the ratio of the

successive derivatives until a non-indeterminate form is found. For example,

$$\lim_{x\to 0}\frac{\sin 2x - 2\sin x}{x^3} = \frac{0}{0} = \lim_{x\to 0}\frac{2\cos 2x - 2\cos x}{3x^2} = \frac{0}{0}$$
$$= \lim_{x\to 0}\frac{-4\sin 2x + 2\sin x}{6x} = \frac{0}{0} = \lim_{x\to 0}\frac{-8\cos 2x + 2\cos x}{6} = -1$$

De l'Hospital's rule also applies when both $y_2(x)$ and $y_1(x)$ approach infinity as x approaches x_0.

$$\lim_{x\to x_0}\frac{y_2(x)}{y_1(x)} = \frac{\infty}{\infty} = \frac{y_2'}{y_1'}\bigg]_{x=x_0}$$

Table 1·7 gives a résumé of the indeterminate forms encountered in engineering computations and indicates how they can be reduced to forms that can be evaluated by de l'Hospital's rule.

TABLE 1·7

$f = 0$	$\varphi = 0$	$\frac{f}{\varphi} = \frac{0}{0} = \frac{f'}{\varphi'}$	(a)
$f = \infty$	$\varphi = \infty$	$\frac{f}{\varphi} = \frac{\infty}{\infty} = \frac{f'}{\varphi'}$	(b)
$f = 0$	$\varphi = \infty$	$f\varphi = 0\cdot\infty = \frac{f}{1/\varphi} = \frac{0}{0}$	(c)
$f = \infty$	$\varphi = 0$	$f\varphi = \infty\cdot 0 = \frac{f}{1/\varphi} = \frac{\infty}{\infty}$	(d)
$f = \infty$	$\varphi = \infty$	$f - \varphi = \infty - \infty = f\left(1 - \frac{\varphi}{f}\right)$	(e)
	If $\frac{\varphi}{f} = \infty$	$f - \varphi = -\infty$	(e1)
	If $\frac{\varphi}{f} = 0$	$f - \varphi = \infty$	(e2)
	If $\frac{\varphi}{f} = 1$	see Eq. (d)	(e3)
	If $\frac{\varphi}{f} = k \lessgtr 1$	$f - \varphi = \pm\infty$	(e4)

$f = 0$	$\varphi = 0$	$f^\varphi = 0^0$	$\log f^\varphi = \varphi \log f = 0(-\infty)$	(f)
$f = \infty$	$\varphi = 0$	$f^\varphi = \infty^0$	$\log f^\varphi = \varphi \log f = 0\cdot\infty$	(g)
$f = 1$	$\varphi = \infty$	$f^\varphi = 1^\infty$	$\log f^\varphi = \varphi \log f = \infty\cdot 0$	(h)

For Eqs. (f), (g), and (h), use Eq. (d).

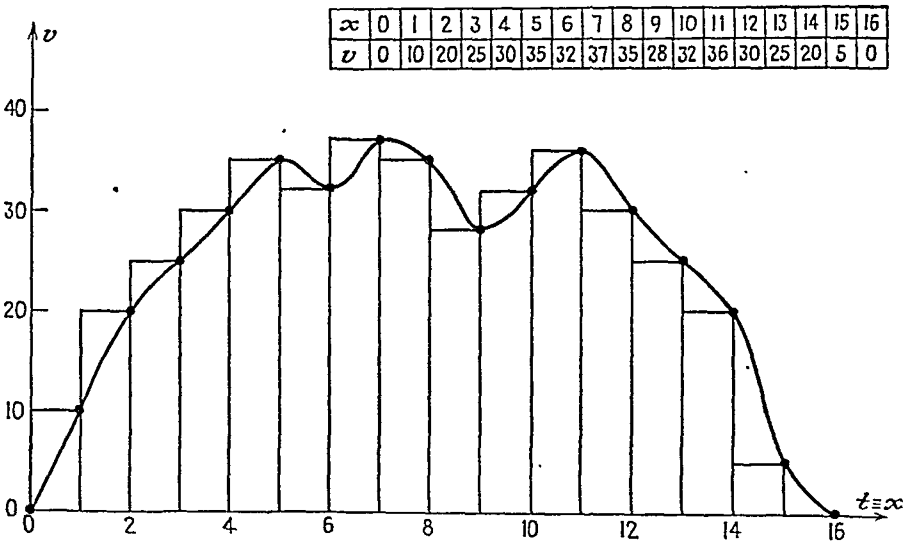
The most frequently encountered indeterminate forms are 0/0 and ∞/∞.

1·9 Integrals

a. *The Definite Integral*

The following simple procedure can be used to check the accuracy of the speedometer readings of a car over a certain range of speeds: The

car is driven from New York to Yonkers, and speedometer readings are taken at equal intervals of time Δt, say every minute ($\frac{1}{60}$ hr). These readings are plotted in Fig. 1·17. If during the first minute the speed of the car did not change too much, the mileage covered during the first minute would be approximately equal to $v_1\,\Delta t$, where v_1 is the speed at the end of the first minute, and would be measured by the area of the first rectangle in Fig. 1·17. Similarly, the mileage in the ith minute is approximately equal to the area of the corresponding rectangle $v_i\,\Delta t$, v being the speed at the end of the ith minute. Hence the sum of the



x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
v	0	10	20	25	30	35	32	37	35	28	32	36	30	25	20	5	0

FIG. 1·17.

areas of all the rectangles of Fig. 1·17 is a rough estimate of the mileage covered in 16 min. The distance between New York and Yonkers being known, a comparison between the actual and the computed mileages allows an over-all check on the speedometer reading over the range of speeds used in the test.

If the velocities v are given in miles per hour, the mileage after 16 min computed from Fig. 1·17 is

$$m_{16} = v_1 \cdot \frac{1}{60} + v_2 \cdot \frac{1}{60} + \cdots + v_{16} \cdot \frac{1}{60}$$

or, using the symbol Σ (capital sigma) to indicate the sum of the terms in the right-hand member of this equation,

$$m_{16} \doteq \sum_{i=1}^{16} v_i\,\Delta t$$

Under the assumption that the speed remains constant during any given minute, the area under the curve $v(t)$ was approximated in this check by the sum of the areas of the rectangles of width $\Delta t = 1/60$ hr. The accuracy of the check may be improved by taking readings every 30 sec, *i.e.*, making $\Delta t = 1/120$ hr, since the speed can be kept more nearly constant during 30 sec than during a full minute. The computed mileage then becomes

$$m_{16} \doteq v_1 \cdot \frac{1}{120} + v_2 \cdot \frac{1}{120} + \cdots + v_{32} \cdot \frac{1}{120} = \sum_{i=1}^{32} v_i \, \Delta t$$

Geometrically, we have doubled the number of rectangles in Fig. 1·17 by cutting their width in half.

If now Δt could be made an infinitesimal, the number of rectangles would increase indefinitely, while their width would approach zero and the expression for the mileage would be rigorously correct.

$$m_{16} = \lim_{\Delta t \to 0} \sum_{t=0}^{t=16/60} v(t) \, \Delta t \qquad (a)$$

In this equation the sum is extended, so to speak, to all the values of t in the interval $t = 0$, $t = 16/60$ hr. The following special symbol is used in mathematics for the infinite sum (a):

$$m_{16} = \int_0^{16/60} v(t) \, dt \qquad (1\cdot9\cdot1)$$

and is called the *definite integral* of the function $v(t)$ between the lower limit 0 and the upper limit $16/60$. [The integral symbol is a modification of the letter S, which was originally used to indicate the sum of terms (a).]

The definite integral is a *number*—in our case the mileage after 16 min, in the general case the area under a curve between two given ordinates. If we had chosen to call the time x rather than t, the integral would not be changed in value, since the "name" of the variable does not appear in the result. Hence

$$\int_0^{16/60} v(t) \, dt = \int_0^{16/60} v(x) \, dx = \int_0^{16/60} v(L) \, dL \qquad (1\cdot9\cdot2)$$

t, x, L are different names for the *variable of integration* appearing under the integral sign, and Eq. (1·9·2) shows that a definite integral is *not* a function of the variable of integration. This variable is sometimes called a "dumb" variable.

b. Integral with Variable Limits

From the graph of Fig. 1·17 we may also compute the mileage after, say 10 min., which is given by the integral

$$m_{10} = \int_0^{10/60} v(x)\,dx$$

or the mileage $m(t)$ after t hr, which is given by

$$m(t) = \int_0^t v(x)\,dx \tag{1·9·3}$$

The integral $m(t)$ is now a *function* (not a number); it is *a function of the upper limit t, not of the variable of integration x.* Unfortunately, the *integral with upper variable limit* [Eq. (1·9·3)] is often written

$$m(t) = \int_0^t v(t)\,dt$$

confusing the variable upper limit with the variable of integration. This confusion should always be avoided.

c. The Integral as an Anti-differential

A fundamental property of the integral [Eq. (1·9·3)] with an upper variable limit t will now be proved by differentiating it with respect to this limit.

Using the Δ method (Sec. 1·7 *a,b*), we find

$$m(t) = \int_0^t v(x)\,dx$$

$$m(t + \Delta t) = \int_0^{t+\Delta t} v(x)\,dx$$

$$\Delta m = m(t + \Delta t) - m(t) = \int_0^{t+\Delta t} v(x)\,dx - \int_0^t v(x)\,dx$$

$$\frac{\Delta m}{\Delta t} = \frac{1}{\Delta t}\left[\int_0^{t+\Delta t} v(x)\,dx - \int_0^t v(x)\,dx\right]$$

$$\frac{dm}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_0^{t+\Delta t} v(x)\,dx - \int_0^t v(x)\,dx\right] = \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} \tag{a}$$

By Eq. (1·9·3) the first integral in the right-hand member of Eq. (*a*) is the area under the $v(t)$ curve between 0 and $t + \Delta t$; the second integral is the area under the same curve between 0 and t (see Fig. 1·18). Hence the quantity Δm is the area of the strip $BCDE$, shown enlarged in Fig. 1·19, which is certainly larger than or equal to the area $v(t) \cdot \Delta t$ of the rectangle $BCFE$ and smaller than or equal to the area $v(t + \Delta t) \cdot \Delta t$ of

the rectangle $BGDE$. Therefore the quantity $\Delta m/\Delta t$ in the right-hand member of Eq. (a) is bounded as follows:

$$v(t) \leq \frac{\Delta m}{\Delta t} \leq v(t + \Delta t) \tag{b}$$

the equal signs representing the particular cases when $v(t)$ is constant between t and $t + \Delta t$. Taking now the limit of the three terms of Eq. (b) as Δt approaches zero, and noting that $\lim_{\Delta t \to 0} v(t + \Delta t) = v(t)$, we find that

$$v(t) \leq \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} = \frac{dm}{dt} \leq v(t)$$

or, since a quantity cannot be at the same time smaller and larger than

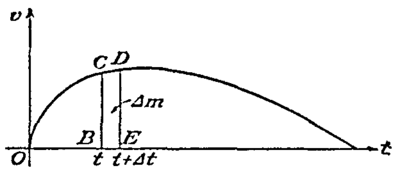


Fig. 1·18.

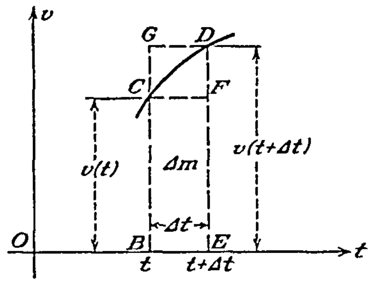


Fig. 1·19.

another,

$$\frac{dm}{dt} = v(t) \tag{1·9·4}$$

In words, the derivative of an integral *with upper variable limit* with respect to this limit is equal to the function under the integral sign *taken at the upper limit.*

The function $m(t)$, limit of a sum and area under the curve $v(t)$, *can now be looked upon as being the function whose derivative equals* $v(t)$.

This entirely different interpretation of the integral with variable upper limit is of fundamental importance in the applications of the calculus and was known to its discoverers, Newton and Leibnitz.

From Eq. (1·9·4),

$$dm = v(t)\,dt$$

and, integrating both sides between 0 and t,

$$\int_0^t dm = \int_0^t v(t)\,dt = \int_0^t v(x)\,dx = m(t)$$

by Eq. (1·9·3). This result shows how the operation of integration is

the inverse of the d operation ("take the differential"), since, applying to $m(t)$ first the d operation and then integration, we obtain m again. (Similarly, the operation "squaring" is called the inverse of the operation "take the square root," for the square of the square root of a number is the same number.)

What has been said of integrals with an *upper* variable limit can be extended to integrals with a lower variable limit by noticing that

$$\int_a^a v(x)\,dx = 0 = \int_a^t v(x)\,dx + \int_t^a v(x)\,dx \quad (a = \text{const})$$

and hence

$$\int_t^a v(x)\,dx = -\int_a^t v(x)\,dx \tag{1·9·5}$$

d. Indefinite Integral

We have seen how the integral

$$m(t) = \int_0^t v(x)\,dx$$

is a function whose derivative equals $v(t)$. But since the derivative of a constant is zero, all the functions of the family

$$V(t) = m(t) + C$$

where C is an arbitrary constant, have also a derivative equal to $v(t)$. This *family of functions* is called the *indefinite integral of* $v(t)$ and is represented mathematically by the symbol

$$V(t) = \int v(t)\,dt = m(t) + C \tag{1·9·6}$$

$m(t)$ is the particular function of the family $V(t)$ which becomes zero at $t = 0$, as can be seen by Eq. (1·9·3).

The computation of the numerical value of a definite integral is usually performed by means of the indefinite integral $V(t)$ as follows:

$$\begin{aligned}\int_a^b v(t)\,dt &= \int_0^b v(t)\,dt - \int_0^a v(t)\,dt \\ &= m(b) - m(a) = [m(b) + C] - [m(a) + C] \\ &= V(b) - V(a)\end{aligned}$$

Indicating the difference between $V(b)$ and $V(a)$ by the symbol $V(t)\Big]_a^b$, the procedure for the evaluation of the definite integral is represented by the formula

$$\int_a^b v(t)\,dt = V(t)\Big]_a^b \tag{1·9·7}$$

In words, to compute the definite integral of a function, determine its indefinite integral, and subtract from the value of the indefinite integral at the upper limit the value of the indefinite integral at the lower limit.

e. Techniques of Integration

The definition of the derivative of a function gives at the same time a definite procedure for the computation of derivatives (the Δ method). Moreover, the derivative of an elementary function is always another elementary function.

There are no standard procedures for the computation of the indefinite integral of a given function. Considering integration as the inverse of the d operation, it can well be said that the art of integration consists in a good memory of differentiation. Moreover, integration may lead from known to unknown functions. Thus the function

$$y = \frac{1}{\sqrt{1 - k^2 \sin^2 x}} \qquad (k = \text{const. less than } 1)$$

has an integral

$$F(x,k) = \int_0^x \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}$$

which is a new nonelementary function of x (and of k), called an *elliptic integral of the first kind.* The function $F(x,k)$ is *not* representable by a finite combination of elementary functions.

For this reason the engineer must be familiar with tables of integrals, like Peirce's "A Short Table of Integrals"[1] and Bierens de Haan's "Table of Definite Integrals,"[2] and should memorize the integrals of the elementary functions, given in Table 1·8.

TABLE 1·8

1. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C (n \neq -1)$	2. $\int \frac{1}{x} \, dx = \log_e x + C$
3. $\int \sin x \, dx = -\cos x + C$	4. $\int \cos x \, dx = \sin x + C$
5. $\int \tan x \, dx = \sec^2 x + C$	6. $\int e^x \, dx = e^x + C$
7. $\int e^{-x} \, dx = -e^{-x} + C$	8. $\int \sinh x \, dx = \cosh x + C$
9. $\int \cosh x \, dx = \sinh x + C$	10. $\int \log x \, dx = x \log x - x + C$

Among the techniques of integration often used in elementary derivations, and hence of practical value to the engineer, is *the method of integration by parts.* The formula for integration by parts is based upon the formula for the differentiation of the product of two functions [Eq. (1·7·2)],

$$\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

[1] Ginn & Company, Boston.

[2] G. E. Stechert & Company, New York

Multiplying by dx both sides of this equation and integrating, we obtain

$$\int d(uv) = uv = \int u\,dv + \int v\,du$$

from which

$$\int u\,dv = uv - \int v\,du \tag{1·9·8}$$

This formula is helpful whenever $\int v\,du$ is easier to compute than $\int u\,dv$. For instance, the integral

$$\int_{-\pi}^{\pi} x \cos 2x\,dx$$

can be integrated by parts, remembering that

$$d(2x) = \frac{d(2x)}{dx}\,dx = 2dx$$

$$\cos 2x\,d(2x) = d(\sin 2x)$$

and, taking

$$u = x \qquad v = \sin 2x$$

$$\int_{-\pi}^{\pi} x \cos 2x\,dx = \tfrac{1}{2}\int_{-\pi}^{\pi} x \cos 2x\,d(2x) = \tfrac{1}{2}\int_{-\pi}^{\pi} x\,d(\sin 2x)$$
$$= \tfrac{1}{2}x \sin 2x\Big]_{-\pi}^{\pi} - \tfrac{1}{2}\int_{-\pi}^{\pi} \sin 2x\,dx = -\tfrac{1}{4}\int_{-\pi}^{\pi} \sin 2x\,d(2x)$$
$$= \tfrac{1}{4}\int_{-\pi}^{\pi} d(\cos 2x) = \tfrac{1}{4}\cos 2x\Big]_{-\pi}^{\pi} = \tfrac{1}{4}(1 - 1) = 0$$

Integration by parts leads sometimes to the evaluation of an integral by two successive applications of Eq. (1·9·8) as shown in the following example:

$$\int e^x \sin x\,dx = \int e^x\,d(-\cos x) = -e^x \cos x + \int e^x \cos x\,dx$$
$$= -e^x \cos x + \int e^x\,d(\sin x) = -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx$$

It will be noticed that the last integral is equal to the integral we are trying to evaluate. Transposing it to the left-hand member, we get

$$2\int e^x \sin x\,dx = e^x(\sin x - \cos x)$$

from which

$$\int e^x \sin x\,dx = \tfrac{1}{2}e^x(\sin x - \cos x) + C$$

where C is a constant of integration.

It is well to remember that, while the process of integration may be complicated, the results of integration can always be checked by differentiation.

f. Numerical Integration

When a function is not known analytically or cannot be integrated by elementary methods, its definite integral can always be obtained by

numerical methods. One such method was outlined in Sec. 1·9*a*, but more accurate procedures have been devised, among which *Simpson's rule* is perhaps the most practical.

Table 1·9 gives the value of the velocity y in miles per hour of the car of Sec. 1·9 *a* versus time x in minutes. In order to compute the mileage

TABLE 1·9

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
y	0	10	20	25	30	35	32	37	35	28	32	36	30	25	20	5	0

covered in 16 min., *i.e.*, the area under the curve y between $x = 0$ and $x = 16$, let us consider three consecutive points, A, B, and C, on this curve (Fig. 1·20) whose abscissas a, b, and c are equally spaced by an amount Δx. Simpson's rule consists in substituting for the actual curve $y(x)$ in the interval a,c a quadratic parabola

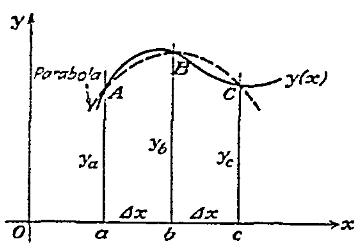


FIG. 1·20

$$y(x) = Lx^2 + Mx + N \qquad (a)$$

passing through the three points A, B, and C, and in computing the area under this parabola rather than the area under the curve. The area under the parabola (a) between a and c is given by

$$S_{a,c} = \int_a^c (Lx^2 + Mx + N)\,dx = \frac{Lx^3}{3} + \frac{Mx^2}{2} + Nx\bigg]_a^c$$

$$= L\frac{c^3 - a^3}{3} + M\frac{c^2 - a^2}{2} + N(c - a) \qquad (b)$$

If it were necessary to know beforehand the coefficients L, M, N of the parabola in order to compute the area $S_{a,c}$, Simpson's rule would be impractical. But, as we shall now prove, this area can instead be easily computed by means of the ordinates y_a, y_b, and y_c of the curve. To this end, we notice that, since the quadratic parabola goes through the points A, B, and C, and since b is the middle point of the segment a,c [$b = \frac{1}{2}(a + c)$],

$$y_a = La^2 + Ma + N$$
$$y_b = \tfrac{1}{4}L(a + c)^2 + \tfrac{1}{2}M(a + c) + N$$
$$y_c = Lc^2 + Mc + N$$

Let us compute the following expression involving the ordinates y_a, y_b and y_c:

$$\frac{c-a}{6}(y_a + 4y_b + y_c) = \frac{c-a}{6}\left\{(La^2 + Ma + N) + 4\left[\frac{1}{4}L(a+c)^2 + \frac{1}{2}M(a+c) + N\right] + (Lc^2 + Mc + N)\right\}$$

$$= (c-a)\left[\frac{1}{3}L(a^2 + ac + c^2) + \frac{1}{2}M(a+c) + N\right]$$

$$= L\frac{c^3 - a^3}{3} + M\frac{c^2 - a^2}{2} + N(c-a) \qquad (c)$$

Comparison of Eqs. (*b*) and (*c*) shows that

$$S_{a,c} = \frac{c-a}{6}(y_a + 4y_b + y_c)$$

or, since $c - a = 2\Delta x$, that

$$S_{a,c} = \frac{\Delta x}{3}(y_a + 4y_b + y_c) \qquad (d)$$

Upon dividing the interval $x_0 = 0$, $x_n = 16$ into an *even* number n of subintervals of width $\Delta x = (x_n - x_0)/n$, Eq. (*d*) can be successively applied to each couple of subintervals, and the area A under the curve becomes

$$A = \frac{\Delta x}{3}[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + (y_4 + 4y_5 + y_6) + \cdots + (y_{n-2} + 4y_{n-1} + y_n)$$

or

$$A = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n) \qquad (1·9·9)$$

Equation (1·9·9) is called *Simpson's formula.*

In order to obtain an accurate evaluation of an integral by Eq. (1·9·9) it is advisable to start with a small number of strips and to double this number in successive computations. In the case of Table 1·9 we can start with $n = 2$, that is, with $\Delta x = (16 - 0)/2 = 8$ min $= 8/60$ hr.

$$A_2 = \frac{8/60}{3}(0 + 4 \times 35 + 0) = 6.22 \text{ miles}$$

For $n = 4$ we have similarly

$$A_4 = \frac{4/60}{3}(0 + 4 \times 30 + 2 \times 35 + 4 \times 30 + 0) = 6.88 \text{ miles}$$

For $n = 8$ and $n = 16$ the computations are conveniently arranged in tabular form (see Table 1·10).

The accuracy of the result can be indefinitely improved by increasing the number of points considered, but two successive approximations are sufficient to evaluate the order of magnitude of the error in the second approximation. In fact, it can be proved that the error e in the approximate value obtained by using n subintervals is roughly proportional to $1/n^4$:

$$e \doteq Kn^{-4}$$

TABLE 1·10.—TABULAR COMPUTATIONS BY SIMPSON'S RULE
$n = 8 \quad \Delta x = 2/60; \quad n = 16 \quad \Delta x = 1/60$

x	y	M	My	M	My
0	0	1	0	1	0
1	10	...	...	4	40
2	20	4	80	2	40
3	25	..	...	4	100
4	30	2	60	2	60
5	35	. .	...	4	140
6	32	4	128	2	64
7	37	.	...	4	148
8	35	2	70	2	70
9	28	..	...	4	112
10	32	4	128	2	64
11	36	.	..	4	144
12	30	2	60	2	60
13	25		...	4	100
14	20	4	80	2	40
15	5		...	4	20
16	0	1	0	1	0
			$\Sigma = 606$		$\Sigma = 1202$

$$A_8 = \frac{2/60}{3}\,606 = 6.73 \qquad A_{16} = \frac{1/60}{3}\,1202 = 6.68$$

Hence, if we call n_i and n_j the number of subintervals used in two successive approximations, the ratio of the corresponding errors e_i and e_j becomes

$$\frac{e_i}{e_j} = \frac{n_j^4}{n_i^4}$$

from which

$$e_i = \left(\frac{n_j}{n_i}\right)^4 e_j$$

The error being the difference between the true value A and the approximate value A_n,

$$e = A - A_n$$

we can write, for the two approximations A_{n_i} and A_{n_j},

$$A = A_{n_i} + e_i \doteq A_{n_i} + \left(\frac{n_j}{n_i}\right)^4 e_j$$

$$A = A_{n_j} + e_j$$

and, subtracting the second from the first of these equations,

$$A - A = 0 \doteq A_{n_i} - A_{n_j} + \left[\left(\frac{n_j}{n_i}\right)^4 - 1\right] e_j$$

from which the error in the last approximation becomes

$$e_j \doteq \frac{A_{n_j} - A_{n_i}}{(n_j/n_i)^4 - 1} \tag{1·9·10}$$

When the number of subintervals is doubled from one approximation to the next, $n_j/n_i = 2$ and the error becomes

$$e_j = \tfrac{1}{15}(A_{n_j} - A_{n_i}) \tag{1·9·11}$$

In the previous example,

$$e_{16} = \frac{1}{15}(6.68 - 6.73) = -\frac{0.05}{15} = -0.0033$$

and A_{16} is probably correct within one unit in the second decimal figure.

When the number of subintervals used is odd, say $2m + 1$, the area of the first $2m$ strips is computed by Simpson's rule and the area of the last strip by assuming a straight-line variation of the graph between the last two points, *i.e.*, by computing the area of the corresponding trapezoid

$$S_{2m,\,2m+1} = \tfrac{1}{2}\Delta x(y_{2m} + y_{2m+1})$$

Thus the last approximation of the mileage after 15 min in Table 1·9 is

$$A_{15} = A_{14} + S_{15} = \frac{\frac{1}{60}}{3}1162 + \frac{\frac{1}{60}}{2}(20 + 5) = 6.456 + 0.208 = 6.664$$

By computing the area A for different values of x_n the indefinite integral of a function can also be evaluated by Simpson's rule within a given range of values of x.

Problems

1. State the difference between rational, irrational, imaginary, and complex numbers.

2. To express the length $2\frac{3}{4}$ ft by means of an integer, we use the inch as unit. What unit should be used to express by means of an integer the length $\sqrt{8}$?

3. Construct with ruler and compass the square root of 4 and the square root of 10, starting with a line segment of unit length.

4. Prove that $\sqrt[3]{7}$ is not a rational number.

5. Prove that $\log_{10} 5$ is not a rational number.

6. Classify the following numbers according to the classes of Table 1·1:

(a) $-i$ (b) $-7/3$
(c) e^2 (d) $4 + 0i$
(e) $\sqrt{2}\, i$ (f) $\pi - 7i$
(g) $4/2$ (h) $0.2222 \ldots$
(i) $0 + 2.1i$ (j) $6.714285 \quad 714285 \ldots$

7. Compute the absolute value of the following numbers:

(a) -3 (b) 4.5
(c) $-3 + 2i$ (d) $-3 - 2i$
(e) $\sqrt{2}\, i$ (f) $2.4 + 1.2i$

8. Express in the form $a + bi$ the following complex numbers

(a) $\dfrac{1}{1 - i}$ (b) $\dfrac{3}{1 + \sqrt{3i}}$
(c) $\dfrac{5}{2 + 7i}$ (d) $\dfrac{1 - i}{1 + 2i}$
(e) $\dfrac{5 - 5i}{10 + 10i}$ (f) $\left(\dfrac{4}{3 - 2i}\right)^{-1}$
(g) $(7 + 4i)^2$ (h) $(-2 + 3i)^3$
(i) $\dfrac{1}{(4 + 2i)^3}$ (j) $\dfrac{2 + 7i}{(7 - 2i)^2}$

9. Perform analytically and check graphically the following operations:

(a) $(4 - 2i) + (3 + i)$ (b) $(7 + 3i) - (6 - 3i)$
(c) $(i) + (2 + 4i)$ (d) $(3a + 6ai) + (4a^2)$
(e) $\dfrac{7}{1 + 3i}$ (f) $\dfrac{4 + 2i}{6 - 3i}$
(g) $(7 + 3i)(2 - i)$ (h) $(6 + 4i)(2i)$
(i) i^2, i^3, i^4 (j) $(3 + 2i)^2$

10. Find the complex conjugates of the following numbers:

(a) $2 - 3i$ (b) $-2 - 4i$
(c) 9 (d) -4
(e) $2i$ (f) $-\sqrt{2}\, i$

11. Express the following numbers in polar form:

(a) $4 + 3i$ (b) $7 + 2i$
(c) $-5 + 6i$ (d) $0.001 - 0.0015i$
(e) $-0.7 - 0.8i$ (f) $1 - 5i$
(g) 2.5 (h) $-0.01i$
(i) $10^{-4} - 8.5 \times 10^{-5}i$ (j) $2 \times 10^5 - 4 \times 10^5 i$

12. Express the following numbers in Cartesian form:

(a) $12(\cos 0.4 + i \sin 0.4)$ (b) $0.3[\cos(-0.7) + i \sin(-0.7)]$
(c) $4.3(\cos 10 + i \sin 10)$ (d) $7(\cos 40° + i \sin 40°)$
(e) $4 \times 10^{-3}(\cos 2 \times 10^{-2} + i \sin 2 \times 10^{-2})$
(f) $2.25(\cos 32°10' + i \sin 32°10')$

Note: Angles are measured in radians unless otherwise stated.

13. Perform the following operations by means of the polar form of the numbers:

(*a*) $(2\underline{/30^\circ} \times 3\underline{/25^\circ})$
(*b*) $2.45\underline{/31^\circ 15'} \div 12.2\underline{/10^\circ 25'}$
(*c*) $(3.2\underline{/20^\circ})^2$
(*d*) $(16.2\underline{/272^\circ})^{-3}$
(*e*) $(3.15\underline{/24^\circ})^2 \times (6.12\underline{/224^\circ})^{-4}$
(*f*) $(r\underline{/\theta}) \times (r\underline{/-\theta})$
(*g*) $(1\underline{/0}) \times (r\underline{/\theta})^{-1}$
(*h*) $(r\underline{/\theta}) \times \left(r\underline{\Big/ -\theta + \frac{\pi}{2}}\right)$

14. Compute analytically and check by means of a polar diagram all the values of the indicated roots.

(*a*) $(40 + 30i)^{1/2}$
(*b*) $\sqrt[3]{14}$
(*c*) $(7 - 2i)^{1/4}$
(*d*) $(16i)^{1/4}$
(*e*) $(20 + 3i)^{1/3}$
(*f*) $(20\underline{/0.8})^{1/4}$
(*g*) $(16 + 0i)^{1/2}$
(*h*) $(-19 - 14i)^{1/2}$
(*i*) $\sqrt[3]{(7 - 13i)^2}$
(*j*) $(0.232 - 0.042i)^{1/5}$

15. Find all the roots of the following algebraic equations:

(*a*) $x^3 - 1 = 0$
(*b*) $x^4 = 2$
(*c*) $2x^7 + 7 = 0$
(*d*) $x^2 + 3i = 0$
(*e*) $x^3 - i = 0$
(*f*) $x^6 + 1 = 0$

16. Find all the pairs of numbers z_1, z_2 such that

$$z_1^2 = z_2 \qquad \text{and} \qquad z_2^2 = z_1$$

17. Verify by direct calculation that

$$z^3 = r^3(\cos 3\theta + i \sin 3\theta)$$

where $z = r(\cos \theta + i \sin \theta)$.

18. Show that the rational numbers are everywhere dense, *i.e.*, that between any 2 rational numbers there always falls another rational number.

19. In an electric circuit the current I is 5 amp and lags the voltage E of 100 volts by 22.5 deg. What is the impedence Z of the circuit? *Hint:* Represent the complex quantities E, I, and Z by vectors, and remember that $E = ZI$.

20. If a voltage of 240 volts at a frequency of 60 cycles per sec is applied to the circuit below (see Fig. 1·21),

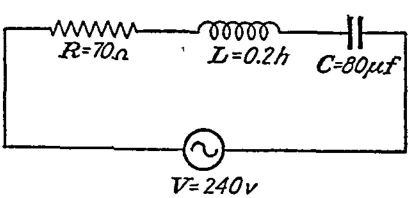


FIG. 1·21.

(*a*) What is the voltage across each element?
(*b*) What is the current through each element?
(*c*) Draw a vector diagram, and show that the sum of the voltages across each element equals the applied voltage. *Hint:* A pure inductance of L henrys has a reactance $= +i\omega L$. A pure capacitance of C farads has a reactance $= -i/\omega C$.

21. Mention some variable quantities depending upon (*a*) the time; (*b*) 1 spatial coordinate; (*c*) 3 spatial coordinates; (*d*) the temperature; (*e*) the time and the temperature; (*f*) the time, the temperature, and 3 spatial coordinates.

22. (*a*) If the friction encountered by a car is proportional to the square of its speed, express the direct and inverse functional relationship between friction and speed. (*b*) The displacement of a particle along a fixed axis from a fixed origin varies

sinusoidally with time. The particle starts from rest at $t = 0$ and has a period of 7 sec. Its maximum displacement from the origin is 4 ft. Express the displacement as a function of time.

23. Mention 3 single-valued and 3 multivalued functions of the real variable x.

24. Specify the field of definition of the following *real* functions:

(a) $\sqrt{36 - x^2}$ (b) $\sin x$
(c) $\log_{10} x$ (d) $e^{\sqrt{x}}$
(e) $\cos \sqrt[4]{81 - x^4}$ (f) $e^{\sqrt{4-x^2}}$
(g) $\log (\sin x)$ (h) $\cosh^{-1} x$

25. Mention 3 composite functions of a variable x through an intermediate variable y, having physical significance.

26. Represent by means of a table and a graph the following functional relationships in the given intervals. Plot 10 values of y versus x.

(a) $y = \frac{1}{10}(x^2 - x + 1)$ $(0 \le x \le 2)$
(b) $y = 0.25(\cos x - \sin x)$ $(0 \le x \le \pi)$
(c) $y = \begin{cases} 0.5x & (0 \le x \le 1) \\ 0.5 + (x - 1)^2 & (1 \le x \le 3) \end{cases}$
(d) $y = \frac{1}{2}(e^x - e^{-x})$ $(-1 \le x \le 1)$
(e) $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ $(-1 \le x \le 1)$
(f) $y = 3x + n$ $(n - 1) \le x \le n$ $(0 \le n \le 3)$

27. State the necessary conditions for a variable x to approach a value x_0 as a limit.

28. Which of the following sequences indicate an approach as a limit? Determine the value of the limit, when the limit exists.

(a) 10, 9, 8.1, 7.29, . .
(b) 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, .
(c) 10, 9, 8, 7, 6, .
(d) 1, $\frac{1}{2}$, $\frac{1}{4}$, 1, $\frac{1}{8}$, $\frac{1}{16}$, 1, $\frac{1}{32}$, $\frac{1}{64}$, 1, . . .
(e) 1, $-\frac{1}{2}$, $+\frac{1}{4}$, $-\frac{1}{8}$, $+\frac{1}{16}$, $-\frac{1}{32}$, . . .
(f) 2, 1.5, 1.25, 1.125, . . .
(g) −8, −4, −2, −1, −0.5, . . . , 1.125, 1.25, 1.50, 2, 3, 5, 9, . . .
(h) 1, 4, 9, 16, 25, . . .
(i) ln 1, ln 2, ln 3, ln 4, . . .
(j) 1, $\frac{1}{8}$, $\frac{1}{27}$, $\frac{1}{64}$, . . .

29. Evaluate the following limits:

(a) $\lim\limits_{x \to 4} (x^2 - 7x + 3)$ (b) $\lim\limits_{x \to 2} \dfrac{x^2}{x + 1}$

(c) $\lim\limits_{x \to 0} \dfrac{x^3}{x^2 + x}$ (d) $\lim\limits_{x \to \infty} \dfrac{x}{x^2 + 1}$

(e) $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3}$ (f) $\lim\limits_{x \to 1} \dfrac{x^2 - 2x + 1}{x - 1}$

30. Specify the points of discontinuity, if any, of the following functions in the given intervals, and sketch their graphs in the given intervals:

(a) $x^2 - 2x + 3$	$(-1 \leq x \leq 1)$
(b) $\sin 3x$	$(-\pi \leq x \leq \pi)$
(c) $\tan 2x$	$(-\pi \leq x \leq \pi)$
(d) $\dfrac{1}{2x - 1}$	$(-1 \leq x \leq 1)$
(e) $\dfrac{2x}{(x - 2)^2}$	$(0 \leq x \leq 3)$
(f) $\dfrac{1}{\cos x}$	$(0 \leq x \leq 2\pi)$
(g) $\log_e x$	$(0 \leq x \leq 3)$
(h) $y = \begin{cases} 3x \\ 5 + (x-1)^2 \end{cases}$	$(0 \leq x \leq 1)$ $(1 \leq x \leq 3)$

31. For what values of x are the following functions infinitesimals?

(a) $(x - 2)^2$
(b) $\dfrac{1}{x}$
(c) $\sqrt{x^2 - 2x + 1}$
(d) $\sqrt[3]{x^3 - 3x^2 + 3x - 1}$
(e) $\cos 3x$
(f) $\ln 2x^2$

32. What is the principal part of the following infinitesimals as x approaches zero?

(a) $x + 12x^2 + \frac{1}{2}x^3$
(b) $\sqrt[3]{x^2 + 12x^5}$

33. Which of the following functions are infinitesimals as x approaches zero? What is the order of the infinitesimal in each case?

(a) x^2
(b) $\dfrac{x^2}{2} - x$
(c) $\sqrt{x^7 - 4x}$
(d) $\cos x^3 - 4x$
(e) $\sqrt[7]{x^3 - 1}$
(f) $\sqrt[5]{(x-1)^2 - 1}$
(g) $\dfrac{x}{x^3}$
(h) $4(x - 1) + (x - 2)^2$
(i) $\dfrac{\sqrt{2x + 4x^3}}{\sqrt[5]{7x^{10} - 4}}$
(j) $e^x - 1$

Hint: Use the series expansion of e^x to determine the order of the infinitesimal of j.

34. Compute the derivatives of the following functions by the Δ method:

(a) $3x^2$
(b) $2x^3 - 7x + 4$
(c) $\cos x$
(d) $\sin x \cos x$
(e) $\dfrac{1}{x}$
(f) x^n
(g) $\dfrac{3x}{x^2 + 2x}$
(h) $\tan x$

35. Find the derivatives of the following functions:

(a) $4.2x^5$
(b) kx^n
(c) $3x^2 - 11x + 5$
(d) $\sqrt{2x}$
(e) $(5x - 2x^2)^6$
(f) $(b - 7x)(a + 3x^2)$
(g) $(1 - x)^{-3}$
(h) $(2 - x)\sqrt{1 + x^2}$
(i) $\dfrac{1 + x^2}{1 - x^2}$
(j) $\sqrt[n]{1 + x^n}$

36. Evaluate the derivatives of the following functions:

(a) a^x (b) $\ln 7x^2$
(c) $\ln \dfrac{1+x}{1-x}$ (d) $\ln \sin x$
(e) $xe^x \sin x$ (f) $\ln (x^3 \tan x)$
(g) $e^x \sin e^x$ (h) $e^{\sin x}$
(i) $\cos (\ln 4x)$ (j) $\tan (e^{3x})$

37. Find the derivatives of the following functions:

(a) $\dfrac{e^{2x} - e^{-2x}}{2}$ (b) $\ln e^x$
(c) $\dfrac{e^{-ax}}{a + 3 \cos ax}$ (d) $e^{-x} \tan x$
(e) $\dfrac{\sin x}{7 - 3 \cos x}$ (f) $b^{4x} \ln 4x$
(g) 6^{3x} (h) $xe^{x \ln x}$
(i) $\dfrac{1}{n} (\ln x)^{1-n}$ (j) $\ln (\ln x)$

38. Compute the derivatives of the following functions:

(a) $\sin^{-1} x$ (b) $\dfrac{1}{\tan^{-1} x}$
(c) $\ln \dfrac{1}{ax + b}$ (d) $x \sec^{-1} x$
(e) $\frac{1}{2} \ln (1 + x^2)$ (f) $\tan^{-1} (e^{-x})$
(g) $\dfrac{e^{ax}(a \sin x - \cos x)}{a^2 + 1}$ (h) $\ln \left(\dfrac{1 - \sqrt{1 - x^2}}{x}\right)$
(i) $\sinh^2 x$ (j) $\ln \tanh 3x$

39. Evaluate the derivatives of the following functions:

(a) $\cosh^2 x^2$ (b) $e^{\sinh x}$
(c) $\tan^{-1} (\sinh x)$ (d) $e^{-x} \cosh x$
(e) $\ln \dfrac{e^x}{1 + e^x}$ (f) $x \csc^{-1} x + \ln (x + \sqrt{x^2 - 1})$
(g) $(\sinh 4x)^3$ (h) $\cosh^2 x - \sinh^2 x$
(i) $-\sqrt{a - 1 + 2x - x^2} + \sin^{-1} \dfrac{x - 1}{\sqrt{a}}$ (j) $\dfrac{1}{2} \tan^{-1} \left[\dfrac{5 \tan (x/2) + 3}{4}\right]$

40. Compute the derivatives of the following functions:

(a) $\csc x$ (b) $\sec^2 x$
(c) $\sqrt{1 - \tan^2 x}$ (d) $\sin^3 \left(\dfrac{\pi}{2} - x\right)$
(e) $\tan x \sec x$ (f) $\sin x \cosh x$
(g) $(\cos^2 x - \sin^2 x)^{\frac{1}{2}}$ (h) $\cot (\sec x - 4)$
(i) $\sec [\sin (x + 3)]$ (j) $\sqrt[3]{\sin^4 x^2}$

41. Evaluate the derivatives of the following functions:

(a) $x^3 + \tan x$ (b) $x \sin x - \cos x$

(c) $x^2 \cos 2x$ (d) $\dfrac{(x+1)\sin 2x}{(x-1)\tan x}$

(e) $(x^2 + 7x)\cos^3 x$ (f) $\dfrac{x^3}{\sin x^3}$

(g) $\sqrt{7x}\,\sqrt[3]{\sec x}$ (h) $\csc^{-1}(x \cos x^2)$

(i) $\dfrac{\tan(7x^2)}{\sin(\sqrt{x}+1)}$ (j) $\dfrac{\sin x^2}{x}$

42. Compute dy/dt for each of the following composite functions:

(a) $y = x^2$ $\quad x = 4t$
(b) $y = \sqrt{1-x}$ $\quad x = \sin^2 t$
(c) $y = \sqrt{x}\sin x$ $\quad x = t^2$
(d) $y = \tan^{-1}(2x)$ $\quad x = 6\sqrt{t-1}$
(e) $y = a^x + 4$ $\quad x = t^3 - 4t$
(f) $y = \ln(x+3)$ $\quad x = \ln t$
(g) $y = e^x$ $\quad x = \ln t$
(h) $y = \sin 2x$ $\quad x = \sin^{-1} t$
(i) $y = \cos e^x - \sin e^x$ $\quad x = \ln 2t$
(j) $y = \dfrac{1+x}{\sqrt{1-x^2}}$ $\quad x = \sin t$

43. If u and v are functions of x and $v \neq 0$, show by the Δ method that

$$\frac{d\left(\dfrac{u}{v}\right)}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

44. Find $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}, \dfrac{\partial^2 z}{\partial x^2}, \dfrac{\partial^2 z}{\partial y^2}, \dfrac{\partial^2 z}{\partial x\,\partial y}$, and verify that $\dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{\partial^2 z}{\partial y\,\partial x}$ for the following functions:

(a) $z = (x+y)^2$ (b) $z = x^3 + y^3 + 3xy$
(c) $z = x^3 + y + e^{x+y}$ (d) $z = \sin(2x+y)$
(e) $z = \tanh(xy)$ (f) $z = \ln(xy)$

45. Evaluate the second derivative of the functions in Prob. 34.

46. If the derivative of a function $f(x)$ is equal to zero, what can you say about $f(x)$?

47. Show that $y = e^x + \sin x + 4$ satisfies the following equation:

$$\frac{d^2y}{dx^2} + y - 4 = 2e^x$$

48. Find 2 functions such that the square of their derivatives equals 1 minus the function itself squared.

49. Find 2 functions such that the square of their derivatives equals 1 plus the function itself squared.

50. What functions $z(x,y)$ have $\dfrac{\partial z}{\partial x} = 4y$ and $\dfrac{\partial z}{\partial y} = 4x$?

51. If $y = f(x)$, prove the following identity:

$$\left(\frac{d^3x}{dy^3}\right)\left(\frac{dy}{dx}\right)^5 + \left(\frac{d^3y}{dx^3}\right)\left(\frac{dy}{dx}\right) = 3\left(\frac{d^2y}{dx^2}\right)^2$$

Hint: Remember that $dy/dx = 1/(dx/dy)$.

52. Put the following statement in mathematical form: "The rate of decomposition of a certain radioactive material at a given time is proportional to the amount of material present at that time. The constant of proportionality is α."

53. Show that, if $F(x,y) = 0$, then

$$\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}}$$

Hint: The function F is a constant, and hence its derivative with respect to x is zero. The relationship $F(x,y) = 0$ defines y as a function of x. Consider F as a function of x both directly and through $y(x)$.

54. If $r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$, verify that $1/r$ satisfies the equation

$$\nabla^2\left(\frac{1}{r}\right) = \frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = 0$$

55. The van der Waals equation for a nonideal gas is

$$P = \frac{RT}{v-b} - \frac{a}{v^2}$$

where P = pressure, in atmospheres
V = volume per mol
T = absolute temperature, in degrees Kelvin
R, a, b = const

Find the rate of change of P with respect to v when v is reduced to half its initial volume v_0.

56. Compute the following expressions to 3 significant figures, considering the increments indicated as very small:

(a) $\sqrt{103} = \sqrt{100 + 3}$ (b) $\sqrt{0.8} = \sqrt{1 - 0.2}$

(c) $\sqrt[3]{350} = \sqrt[3]{343 + 7}$ (d) $\dfrac{1}{\sqrt{82}} = \dfrac{1}{\sqrt{81 + 1}}$

(e) $\dfrac{1}{203} = \dfrac{1}{200 + 3}$ (f) $\sin (45°15') = \sin (45° + 15')$

(g) $\tan 44° = \tan (45° - 1°)$ (h) $\cosh 0.01 = \cosh (0 + 0.01)$

(i) $\ln 1.02 = \ln (1 + 0.02)$ (j) $10^{2.02} = 10^{2+0.02}$

57. The adiabatic expansion of a gas follows the law

$$PV^r = \text{const}$$

where P = pressure
V = volume
r = const

What will be the percentage change in volume at $P = 100$ psi when P is decreased by 0.6 psi? Assume $r = 1.4$.

58. If the radius r of a sphere is increasing at the rate of 0.1 in. per sec,

(a) how fast is the volume changing when $r = 3.1$ ft?
(b) how fast is the surface increasing when $r = 2.3$ ft?

59. Show that, for small angles θ, $\tan \theta \doteq \sin \theta$. *Hint:* $\tan 0 = \sin 0 = 0$. Show that the rates of increase of $\tan \theta$ and $\sin \theta$ are equal around $\theta = 0$.

60. Show that, if ϵ is very small in comparison with unity,

$$(1 + \epsilon)^m \doteq 1 + m\epsilon$$

and that, if η is also small in comparison with unity,

$$(1 + \epsilon)^m(1 + \eta)^n \doteq 1 + m\epsilon + n\eta$$

Hint: For $\epsilon \doteq 0$, $(1 + \epsilon)^m = 1 + m\epsilon$. Considering ϵ as a small increment, show that the rate of change of $(1 + \epsilon)^m$ is the same as the rate of change of $1 + m\epsilon$.

61. Evaluate the following indeterminate forms:

(a) $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x}$ (b) $\lim\limits_{x\to 0} \dfrac{1 - \cos x}{\tan x}$

(c) $\lim\limits_{x\to 0} \dfrac{\ln x}{e^{1/x}}$ (d) $\lim\limits_{x\to \infty} \dfrac{\ln x}{x}$

(e) $\lim\limits_{x\to \infty} \dfrac{x^2}{\sin x}$ (f) $\lim\limits_{x\to 0} x^x$

(g) $\lim\limits_{x\to 0} (1 + ax)^{b/x}$ (h) $\lim\limits_{x\to 0} \left(1 + \dfrac{a}{x}\right)^{bx}$

(i) $\lim\limits_{x\to 0} (\sin x)^{\tan x}$ (j) $\lim\limits_{x\to \infty} \left(\dfrac{e^x}{x} - \dfrac{e^x}{x^2}\right)$

(k) $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x}$ (l) $\lim\limits_{x\to 0} \dfrac{\cosh x - \left(1 + \dfrac{x^2}{2}\right)}{x^4}$

62. Evaluate the following indefinite integrals:

(a) $\int 4x^3\, dx$ (b) $\int \sin 4x\, dx$

(c) $\int \dfrac{dx}{x^2}$ (d) $\int \sinh x \cosh x\, dx$

(e) $\int e^{-x^2}x\, dx$ (f) $\int \dfrac{x + 4}{x^2}\, dx$

(g) $\int \dfrac{dx}{\sqrt{a^2 - x^2}}$ (let $x = a \sin t$) (h) $\int \dfrac{x\, dx}{\sqrt{a^2 - x^2}}$

(i) $\int \dfrac{x^2\, dx}{\sqrt{a^2 - x^2}}$ (integrate by parts, and then let $x = a \sin t$) (j) $\int \sin (\ln x)\, dx$ (let $\ln x = t$)

63. Evaluate the following indefinite integrals.

(a) $\int \sin^{-1} x\, dx$ (b) $\int \sec^2 x\, dx$

(c) $\int \dfrac{(x + 1)^2}{x}\, dx$ (d) $\int \dfrac{(x - 2)\, dx}{x^2 - 3x + 2}$

(e) $\int \cos^2 x\,dx$ (f) $\int \sin(a + bx)\,dx$

(g) $\int \frac{dx}{1 - \cos x}$ (h) $\int \sqrt{2ax - a^2}\,dx$

(i) $\int \frac{dx}{x\sqrt{x^2 - a^2}}$ [let $x = a(\cos t)^{-1}$] (j) $\int \frac{x\,dx}{\sqrt{a + bx}}$ (let $a + bx = t$)

64. Evaluate the following definite integrals:

(a) $\int_0^4 x^3\,dx$ (b) $\int_1^5 \ln x\,dx$

(c) $\int_{-\pi}^{\pi} \sin mx \cos nx\,dx$ (m,n integers) (d) $\int_{-\pi}^{\pi} \sin mx \sin nx\,dx$ (m,n integers; $m = n$, $m \neq n$)

(e) $\int_{-\pi}^{\pi} \cos mx \cos nx\,dx$ (m,n integers; $m = n$, $m \neq n$) (f) $\int_0^{0.5} x^2 \sin x\,dx$

(g) $\int_0^2 \cos 3x\,dx$ (h) $\int_4^3 x \ln x\,dx$

(i) $\int_0^1 \frac{dx}{\sqrt[3]{x + 1}}$ (j) $\int_2^3 \frac{\ln x}{x}\,dx$

65. Evaluate the following indefinite integrals:

(a) $\int xe^{-x} \cos x\,dx$ (b) $\int xe^x \sin x\,dx$

(c) $\int \frac{dx}{1 - \sqrt{x}}$ (let $\sqrt{x} = t$) (d) $\int e^{2x} \cos 4x\,dx$

(e) $\int \tan^2 x\,dx$ (f) $\int e^{3x} \sin 3x\,dx$

66. Evaluate the following integrals, using integration by parts:

(a) $\int_0^t \sin \omega(t - \tau)\,d\tau$ (b) $\int_0^t \tau \sin \omega(t - \tau)\,d\tau$

(c) $\int_0^t \tau \cos \omega(t - \tau)\,d\tau$ (d) $\int_0^t \tau^2 \sin \omega(t - \tau)\,d\tau$

(e) $\int_0^t \tau^2 \cos \omega(t - \tau)\,d\tau$

67. Show that, for n a positive integer,

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

Note: Use de l'Hospital's rule to evaluate $\lim_{x\to\infty} x^n e^{-x}$.

68. Evaluate the following integrals by Simpson's rule, using the indicated number of subintervals:

(a) $\int_0^2 e^{-x^2}\,dx$ $n = 2, 4$

(b) $\int_0^\pi \sin x\,dx$ $n = 4, 8$

(c) $\int_0^1 \sqrt{x} \cos x \, dx$ $\qquad n = 4$

(d) $\int_0^5 \sqrt{25 + x^2} \, dx$ $\qquad n = 5$

(e) $\int_0^3 \sqrt{9 + \sqrt{x}} \, dx$ $\qquad n = 6$

(f) $\int_0^{\pi} \sin \sqrt{x} \, dx$ $\qquad n = 4, 6$

(g) $\int_0^{\pi/2} \sqrt{\cos x} \, dx$ $\qquad n = 2, 4$

Evaluate the error in the second approximation of (a), (b), (f), and (g).

69. Evaluate by Simpson's rule the charge $q = \int_0^t i \, dt$ at $t = 12$ μsec on the plates of a condenser uncharged at $t = 0$, if the current i in the circuit is given by the following table:

t, μsec	0	1	2	3	4	5	6	7	8	9	10	11	12
i, amp	0	0.1	0.3	0.7	1.1	1.5	1.9	2.2	2.5	2.7	2.8	2.9	3.0

70. The temperature of a casting is 1060°F at 12 noon. At 2 P.M. its temperature is observed to be 800°F. If the rate of cooling is proportional to the difference between the temperature of the casting and room temperature (68°F), what will be the temperature of the casting at 5 P.M.?

71. Find the length of the curve $27y^2 = x^3$ from the origin to the point (3, 1).

72. Find the area bounded by the lines $x = 0$, $x = 1$, and $y = 0$ and the curve $xy + y = 1$.

73. Show that the area bounded by a parabola $y = k^2x^2$ and a line $y = c^2$ (which is parallel to the tangent at the vertex of the parabola) equals two-thirds the area of the circumscribed parallelogram.

74. The deflection w at a section x of a simply supported beam of length L, uniformly loaded with a load of q lb per ft, is given by the equation

$$EI \frac{d^2w}{dx^2} = q \frac{x}{2} (L - x)$$

where E is the modulus of elasticity of the beam, I is the moment of inertia, and x is measured from one end of the beam. Compute the deflection at the quarter points, assuming $E = 30 \times 10^6$ psi, $I = 100$ in.^{4}, $q = 1000$ lb per ft, and $L = 30$ ft.

Hint: At the supports ($x = 0$, $x = L$), $w = 0$, and $d^2w/dx^2 = 0$, since the deflection and the bending moment are zero.

75. A closed electric circuit contains a 2-henry coil and a 50-ohm resistance in series. If at $t = 0$ the current through the coil is 10 amp, find the current through the coil at $t = 0.1$ sec. *Note:* The electromotive force (emf) due to the coil is proportional to the rate of change of the current and to the inductance; the emf due to the resistance is proportional to the current and to the resistance. The sum of the emfs in a closed circuit is zero.

76. The instantaneous voltage in a circuit is $e = E \sin 2\pi 60t$. What are (a) its average, (b) its rms (root-mean-square) value over a half cycle in terms of E? *Note:*

The rms of $f(x)$ between a and b is

$$\text{rms} = \sqrt{\frac{1}{b-a}\int_a^b [f(x)]^2\,dx}$$

77. Compute the rms and average values of the voltage in a circuit when the voltage wave form is given by the following graph (see Fig. 1·22), (*a*) over a half cycle, (*b*) over a full cycle. (See note on Prob. 76.)

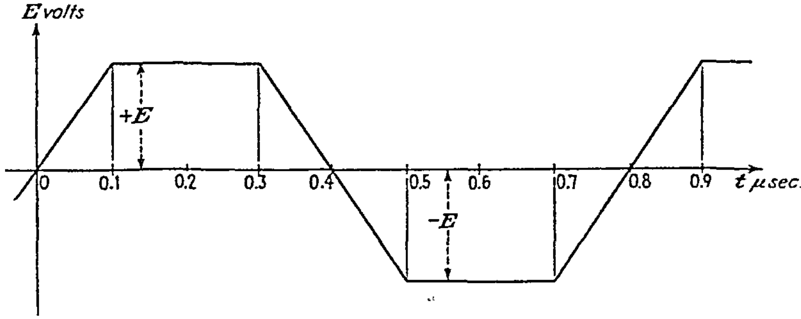


FIG. 1·22.

78. Derive the expression for the work done in compressing isothermally ($PV = K$) a gas from $V = V_1$ to $V = V_2$. *Hint:* Work $= \int_{V_1}^{V_2} P\,dV$.

79. Derive the expression for work done in compressing adiabatically ($PV^r = K$) a gas from $V = V_1$ to $V = V_2$. (See note on Prob. 78.)

80. Deduce the result of Prob. 78 from that of Prob. 79 by taking the limit of the work done adiabatically as r approaches 1.

81. A steam locomotive has the indicator card shown in Fig. 1·23. Compute the mean effective pressure (mep). *Note:* Mep $= \int P\,dV/\int dV$.

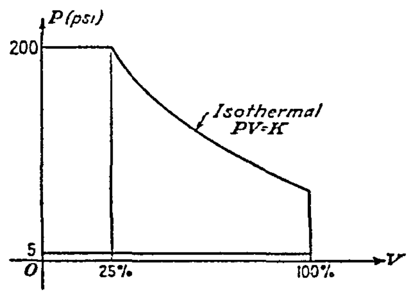


FIG. 1·23.

82. Find the coordinate $\bar{y} = \int y\,ds/\int ds$ of the center of gravity of a semicircle, where ds is an elementary arc of the semicircle.

83. A trough has a parabolic cross section $y = 4x^2$ and is filled with water to a depth H. What is the average pressure exerted on the ends of the trough? *Hint:*

The pressure at a depth h is proportional to the depth of the water and to its specific weight γ.

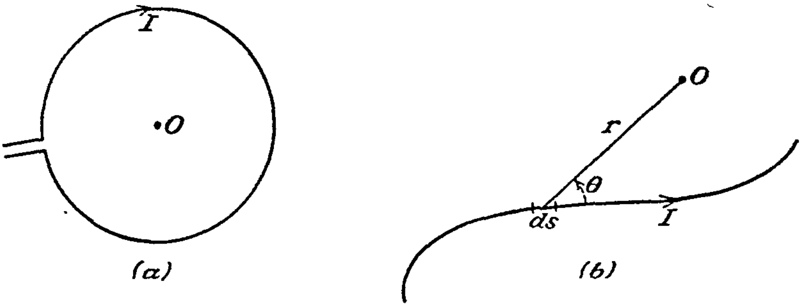


FIG. 1·24.

84. Find the magnetic-field strength H at the center O of a single circular loop of wire carrying a current I. *Hint:* Ampere's law states that $dH = (KI \sin \theta \, ds)/r^2$, where K is a constant depending on the units used and r, θ, and ds are shown in Fig. 1·24b.

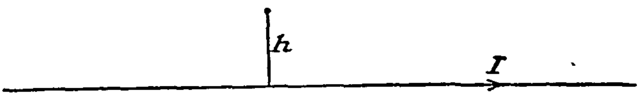


FIG. 1·25.

85. Find the magnetic field at a point a distance h from an infinitely long straight wire carrying a current I. *Note:* See the hint to Prob. 84.

86. A point P moves along the curve $y = x^2/4$ in such a way that its abscissa increases at the rate of 3 units per second. Calling O the origin and Q the foot of the perpendicular from P to the x axis, determine (a) the velocity of P; (b) the rate of change of the area of the triangle OPQ when P is passing through the point (2, 1). *Hint:* Velocity $V = \sqrt{(dx/dt)^2 + (dy/dt)^2}$.

87. A solid has a semicircular base of radius a. The intersections with the solid of each plane perpendicular to the diameter bounding the base is a square. Find the volume of the generated solid.

88. Calculate the work (in ergs) necessary to move a charge of q units through a potential V. *Hint:* $q = CV$, where q is in coulombs, C, the capacitance, in farads, and V the potential in volts.

89. The density of a semicircular thin plate is proportional to the distance from the bounding diameter. If the radius is a and the constant of proportionality is k, find the mass of the semicircle.

90. A wire rope of linear density 0.5 lb per ft and length 1000 ft is suspended in a vertical shaft. Assuming a modulus of elasticity $E = 30 \times 10^6$ psi, what is the maximum stress in the rope, and how much does it elongate owing to its own weight, if its cross section is 0.2 sq in.? *Hint:* The stress σ equals the strain ϵ times E.

91. Prove that

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{a} f(-x)\,dx$$

Hint: Any function $f(x)$ is the sum of an even function $\frac{1}{2}[f(x) + f(-x)]$ and an odd function $\frac{1}{2}[f(x) - f(-x)]$.

92. A concrete wall with a trapezoidal cross section is 20 ft high and has a base 30 ft wide. If its center of gravity G is 9 ft from the ground, what is the angle θ that the sloping side makes with the horizontal (see Fig. 1·26)?

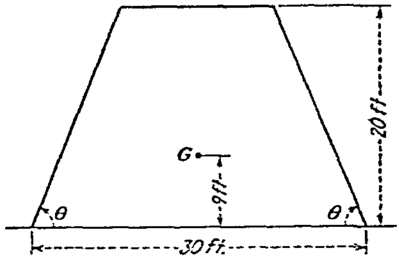


FIG. 1·26.

93. A reconnaissance balloon, finding itself dropping at the rate of 10 ft per sec, throws a sack of sand ballast overboard. If it takes 10 sec for the bag to hit the ground, how high was the balloon when the sand was released? (Neglect air resistance.)

94. If in a certain element of an electric circuit the current (2 amp maximum) leads the voltage (7 volts maximum) by 30 deg, calculate the power being dissipated in the element. *Hint:* the instantaneous voltage and current can be represented as

$$i = 2 \sin (\omega t + 30)$$
$$e = 7 \sin \omega t$$

respectively, and power $= \dfrac{1}{T}\displaystyle\int_0^T ie\,dt$, where $T = \dfrac{2\pi}{\omega}$.

95. If

$$K_1(x,t) = K(x,t)$$

and

$$K_i(x,t) = \int_a^b K(x,s)K_{i-1}(s,t)\,ds$$

show that

$$K_{i+j}(x,t) = \int_a^b K_i(x,s)K_j(s,t)\,ds$$

Hint: Iterate $K_i(x,t) = \displaystyle\int_a^b K(x,s)K_{i-1}(s\ t)\,ds$ j times.

96. The critical load of a simply supported strut of length L, loaded by 2 equal longitudinal compressive forces P, is given by

$$P_{cr} = \frac{\int_0^L EI(y'')^2\,dx}{\int_0^L (y')^2\,dx}$$

where E is Young's modulus, I the moment of inertia and $y(x)$ the deflected shape of the strut. Compute an approximate value of the critical load P for a strut with moment of inertia I from 0 to $L/2$ and $2I$ from $L/2$ to L, assuming $y = A \sin (\pi x/L)$.

97. The mileage recorder of an automobile reads correctly when the tires are new. If after 10,000 indicated miles $\frac{1}{2}$ in. of rubber has worn off each tire, how far has the car actually traveled? Assume that the original diameter of the tires is 2 ft 0 in. and that the decrease in diameter is proportional to the mileage.

98. A certain wall of a furnace is 4 ft high and 6 ft long. Because of inaccuracies in its construction, its width is not constant but varies (linearly) from $8\frac{1}{2}$ to $7\frac{1}{2}$ in. along the length. If the nominal thickness was supposed to be 8 in., what is the increase in heat lost per hour from the furnace to the surrounding atmosphere over that which would have been lost by a correctly designed wall? Assume the temperature of the furnace to be 1068°F, room temperature 68°F, and thermal conductivity of furnace wall 5.0. *Hint:* $Q = \dfrac{KT\,\Delta\theta\,A}{t}$.

where Q = heat, in Btu
T = time, in hours
$\Delta\theta$ = temperature difference, in degrees F
A = area of heat transmission, in square feet
t = thickness, in inches
K = thermal conductivity, in $\dfrac{\text{in.} - \text{Btu}}{°\text{F} - \text{hr} - \text{sq ft}}$

CHAPTER II

PLANE ANALYTIC GEOMETRY

2·1 Coordinates

The publication in 1637 of "La Géométrie" by R. Descartes represents one of the essential steps in the advancement of mathematics. Before Descartes, mathematicians were divided into two separate groups, the geometricians and the algebraists. By noticing that a *point* could be located in a plane by means of two *numbers*, Descartes was able to translate all the algebraic concepts known to him into geometrical concepts and "to perform the marriage between algebra and geometry."

Analytic geometry is an extremely valuable tool of applied mathematics. It allows, on the one hand, the visualization of problems and their graphical solution. On the other, it permits the solution of geometrical problems by the accurate methods of algebra and the calculus. This chapter will be confined to problems of plane geometry and, correspondingly, to functions of one variable.

Uptown New Yorkers live in a Cartesian city. Manhattan is referred to a system of rectangular coordinates in which the number of the house is the abscissa x and the number of the street the ordinate y. All the numbered streets are located in the region of positive y's; positive and negative abscissas are labeled "East" and "West," respectively. The statement: "My address is 210 West Seventy-fifth Street" is translated mathematically into

$$x = -210 \qquad y = +75$$

Since ordinates are measured in blocks and abscissas in buildings, the x and y scales are different. This is usually the case in the system of coordinates used in engineering problems, but it must be emphasized that most of the formulas of this chapter are derived under the assumption of equal x and y scales and do not hold when the scales are different.

A point is located in a plane by means of its two Cartesian coordinates, and we shall write

$$P_1 = P_1(x_1,y_1) \qquad P_2 = P_2(x_2,y_2)$$

2·2 Slopes and Distances

A truck must be driven over a road of constant slope m from a point P_1 to a point P_2, and it must be determined whether or not it can negoti-

ate this slope. Special instruments can be used to measure slopes directly, but the problem is easily solved when the coordinates of the two points are known. Calling y the altitudes and x the distances measured along the road, we derive from the geometry of Fig. 2·1

$$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1} \qquad (2\cdot2\cdot1)$$

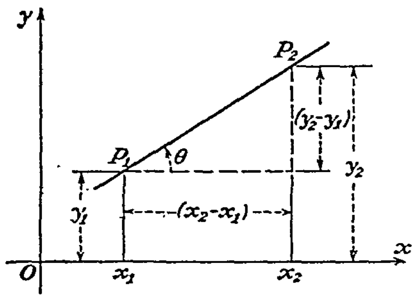


FIG. 2·1.

When two points P_1,P_2 are located by means of Cartesian coordinates, their distance may be easily computed by Pythagoras's theorem. Thus, if, in Fig. 2·1, P_1,P_2 represent now two airports on a map, their distance is given by

$$d = \overline{P_1P_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad (2\cdot2\cdot2)$$

2·3 Straight Lines

A straight road of *constant* slope m must be built starting from a point $P_1(x_1,y_1)$, and a table of altitudes y is needed for points on the road of abscissa x. Since the slope between (x_1,y_1) and *any* point (x,y) on the road is m, we obtain, substituting (x,y) for (x_2,y_2) in Eq. (2·2·1),

$$m = \frac{y - y_1}{x - x_1} \qquad (a)$$

or, multiplying both sides by $(x - x_1)$,

$$y - y_1 = m(x - x_1) \qquad (2\cdot3\cdot1)$$

Transposing y_1, Eq. (2·3·1) may also be written as

$$y = mx + (y_1 - mx_1) \qquad (b)$$

By means of Eq. (*b*) the altitude y of any point of abscissa x can be easily computed, once m, x_1, and y_1 are given.

Since the coordinates of any point lying on the straight line satisfy Eq. (2·3·1) and any two numbers x,y satisfying Eq. (2·3·1) are the coordinates of a point on the line, Eq. (2·3·1) is known as the *equation of the straight line*. It is called the *slope-point* form of the equation of a line because it involves the coordinates of a given point and a given value of the slope. Letting

$$y_1 - mx_1 = b \qquad (c)$$

we may also write Eq. (*b*) as

$$y = mx + b \qquad (2\cdot3\cdot2)$$

Equation (2·3·2) shows that the point of abscissa $x = 0$ on the line has an ordinate equal to b, that is, that b is the length of the segment intercepted by the line on the y axis. For this reason, Eq. (2·3·2) is called the *slope-intercept* form of the line.

A line passing through two *given* points P_1 and P_2 has a slope m given by Eq. (2·2·1). Equating Eqs. (2·2·1) and (a) we find the *equation of a line through two points,*

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}$$

or

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \tag{2·3·3}$$

Equations (2·3·1) to (2·3·3) contain the two coordinates of the variable point $P(x,y)$ *to the first power, i.e.,* they are linear in x and y; any linear equation

$$Ax + By + C = 0 \tag{2·3·4}$$

may be shown to represent a straight line. Dividing Eq. (2·3·4) by B and transposing, we may write

$$y = -\frac{A}{B}x - \frac{C}{B}$$

and comparison with Eq. (2·3·2) shows that the slope and intercept of (2·3·4) are

$$m = -\frac{A}{B} \qquad b = -\frac{C}{B} \tag{2·3·5}$$

A freight train leaves New York at midnight, traveling at a constant speed of 50 mph, and 2 hr later a passenger train leaves New York on the same track, traveling at 70 mph. When and how many miles from New York will the second train overtake the first?

If we measure time in hours starting at midnight and distances in miles from New York, the distance traveled by the first train is

$$S_1 = 50t \tag{d}$$

while the distance traveled by the second train is

$$S_2 = 70(t - 2) \tag{e}$$

The graphs of the two functions S_1 and S_2 are straight lines (as shown in Fig. 2·2), since both S and t appear at the first power in (d) and (e). The second train overtakes the first when the distances S_1 and S_2 are equal for the same value of t, that is, when

$$50t = 70(t - 2)$$

From this equation, $t = 7$ hr; and, by substituting in either Eq. (d) or Eq. (e), $S_1 = S_2 = 350$ miles. It is thus seen that the algebraic solution of this problem is equivalent to the geometrical location of the intersection $P(s,t)$ of the two lines S_1 and S_2. In general, then, the intersection of two lines

$$\left.\begin{aligned} Ax + By + C = 0 \\ A_1x + B_1y + C_1 = 0 \end{aligned}\right\} \qquad (f)$$

is found by solving simultaneously their two equations for the two unknowns x and y.

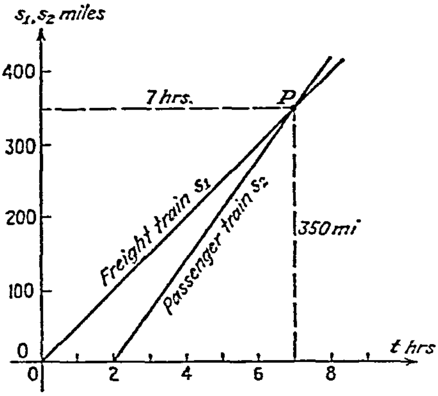


FIG. 2·2.

If the two trains had the same speed, the second would never reach the first and the two lines S_1, S_2, having the same slope, would be parallel. The slopes of the lines (f) being, respectively,

$$m = -\frac{A}{B} \qquad m_1 = -\frac{A_1}{B_1} \qquad (g)$$

the *condition for two lines to be parallel* is

$$\frac{A}{B} = \frac{A_1}{B_1}$$

or

$$\frac{A}{A_1} = \frac{B}{B_1} \qquad (2\cdot3\cdot6)$$

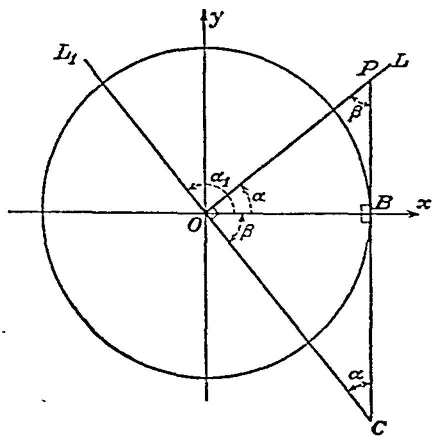


FIG. 2·3.

Two lines are parallel when they have proportional coefficients; the constants C may instead have any values since they simply locate the lines, maintaining them parallel. Very elaborate graphs of the type shown in Fig. 2·2 are used in railroad engineering to set up train schedules and to check the movement of trains.

From the geometry of Fig. 2·3 it is seen that the two triangles OPB and OCB are similar because they have perpendicular sides; hence

$$\frac{BP}{OB} = \frac{OB}{-BC} = \frac{-1}{BC/OB}$$

or

$$\tan \alpha = \frac{-1}{\tan \beta} = \frac{-1}{\tan \alpha_1}$$

Hence, calling m and m_1 the slopes of the two orthogonal lines L and L_1, we find that the relationship between the slopes of two perpendicular lines is

$$m \cdot m_1 = -1 \tag{h}$$

Upon making use of (g), Eq. (h) becomes

$$\frac{A_1}{B_1} = \frac{-B}{A} \tag{i}$$

or

$$AA_1 + BB_1 = 0 \tag{2·3·7}$$

Equation (2·3·7) is called the *condition of orthogonality* of two lines.

Noticing that the angle θ between two lines (Fig. 2·4) is equal to the difference between the angles α_1 and α_2 they make with the x axis and remembering from trigonometry that

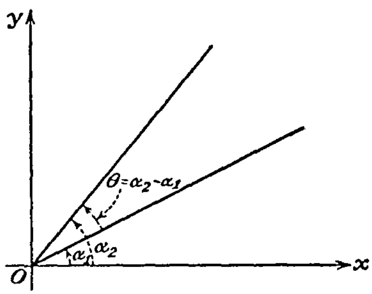


FIG. 2·4.

$$\tan(\alpha_2 - \alpha_1) = \frac{\tan \alpha_2 - \tan \alpha_1}{1 + \tan \alpha_1 \tan \alpha_2}$$

the *angle between two lines* becomes

$$\theta = \arctan \frac{m_2 - m_1}{1 + m_1 m_2} \tag{2·3·8}$$

In particular, when $\theta = 90°$, $\tan \theta = \infty$ and the denominator of Eq. (2·3·8) must equal zero, that is, $m_1 m_2 = -1$, which is Eq. (h).

Given a point $P_0(x_0,y_0)$ and a line

$$Ax + By + C = 0 \tag{j}$$

whose slope, by Eq. (2·3·5), equals $-A/B$, the line through P_0 perpendicular to Eq. (j) becomes, by Eqs. (i) and (2·3·1),

$$y - y_0 = \frac{B}{A}(x - x_0) \tag{k}$$

The coordinates of the intersection $P_1(x_1,y_1)$ of Eqs. (j) and (k) are obtained by solving simultaneously these two equations for x and y and are found to be

$$x_1 = \frac{B^2x_0 - ABy_0 - AC}{A^2 + B^2}$$

$$y_1 = \frac{A^2y_0 - ABx_0 - BC}{A^2 + B^2}$$

The distance d between the point P_0 and the line (j) is therefore given by

$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} = \frac{Ax_0 + By_0 + C}{\sqrt{A^2 + B^2}} \qquad (2\cdot3\cdot9)$$

2·4 Conic Sections

a. The Circle

An airplane starts from an airport O in an unknown direction at a cruising speed of 300 mph and sends an SOS after 20 min ($^{20}\!/_{60}$ hr), indicating that it is forced to land. In what region should search parties look for the crashing airplane?

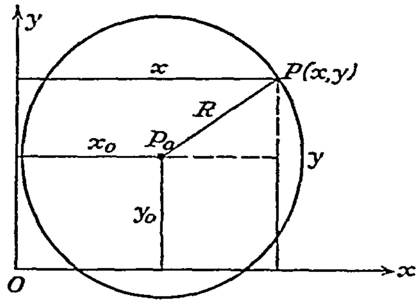


FIG. 2·5.

The distance traveled by the airplane when its SOS is heard equals $300 \times {}^{20}\!/_{60} = 100$ miles; the airplane is therefore inside a circle of 100 miles radius with center at O. To map this circle we notice that, by Pythagoras's theorem, the x,y coordinates of any of its points must satisfy the equation

$$x^2 + y^2 = 100^2$$

since $\sqrt{x^2 + y^2}$ is the distance of $P(x,y)$ from the origin O. This equation is called the *equation of the circle* because it is satisfied by the coordinates of any point on the circle and because any two numbers x,y satisfying it are the coordinates of a point on the circle. Similarly, the equation of a circle of center $P_0(x_0,y_0)$ and radius R (Fig. 2·5) is given by

$$(x - x_0)^2 + (y - y_0)^2 = R^2 \qquad (2\cdot4\cdot1)$$

and contains x and y both to the first and second powers. In order to identify the most general equation of a circle, it is well to remember that the general quadratic equation in two variables,

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0 \qquad (2\cdot4\cdot2)$$

represents a circle if and only if

$$A = B \qquad C = 0 \qquad D^2 + E^2 > 4AF \qquad (2\cdot4\cdot3)$$

The coordinates of the center and the radius of the circle (2·4·2) and (2·4·3) are given by

$$x_0 = -\frac{D}{2A} \qquad y_0 = -\frac{E}{2A} \qquad R = \frac{1}{2A}\sqrt{D^2 + E^2 - 4AF} \qquad (2\cdot4\cdot4)$$

Equation (2·4·2) is the general equation of a "conic section," *i.e.*, of the curve obtained by cutting a circular cone with a plane. The conic section is a circle when the plane is perpendicular to the axis of the cone.

It can be proved that the conic section (2·4·2) degenerates into two straight lines when

$$\begin{vmatrix} 2A & C & D \\ C & 2B & E \\ D & E & 2F \end{vmatrix} = 0 \qquad (a)$$

in which case the cutting plane contains the axis of the cone.

b. The Ellipse

A two-engine airplane leaves from an airport A in an unknown direction after filling its tanks to capacity. Shortly afterward the pilot signals

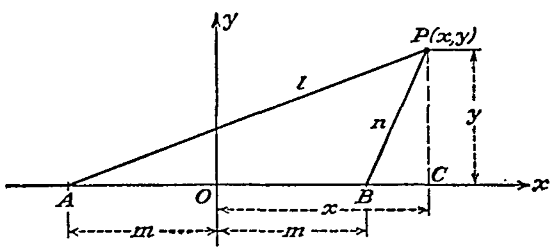


FIG. 2·6.

that one of the engines has gone dead but that he has just enough gasoline to reach airport B, located $2m$ miles from A (Fig. 2·6). Immediately after, the pilot signals again that the airplane is aflame and forced to land. The only additional information available to the airport authorities is that with full tanks that type of airplane can travel d miles and that consumption per mile is roughly equal whether one or two engines are running. Where should the search parties look for the lost airplane?

If at the moment of the forced landing the airplane could still travel a distance of, say, n miles, n must be less than the airplane's distance l from airport A, or the pilot, rather than trying to reach B, would have gone back to A. However, if the airplane had already traveled l miles and could travel only n additional miles,

$$l + n = d$$

since d is the maximum mileage the airplane can travel without refueling.

If the line AB is taken as the x axis with origin at the middle point of the segment AB, the coordinates x,y of a possible landing point P must

be such that

$$AP + PB = l + n = d$$

But, from the geometry of Fig. 2·6,

$$AP = \sqrt{\overline{AC}^2 + \overline{CP}^2} = \sqrt{(x + m)^2 + y^2}$$
$$PB = \sqrt{\overline{BC}^2 + \overline{CP}^2} = \sqrt{(x - m)^2 + y^2}$$

and hence

$$\sqrt{(x + m)^2 + y^2} + \sqrt{(x - m)^2 + y^2} = d$$

Transposing the second radical and squaring, we get, after simplification,

$$d^2 - 4mx = 2d\sqrt{(x - m)^2 + y^2}$$

and, squaring again, simplifying, and dividing by $d^2(d^2 - 4m^2)$,

$$\frac{x^2}{(d/2)^2} + \frac{y^2}{(d/2)^2 - m^2} = 1 \qquad (a)$$

Since $l + n = d > 2m$, $d/2$ is bigger than m and the denominator of the second fraction in Eq. (a) is greater than zero; hence Eq. (a) is of the type

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (2\cdot4\cdot5)$$

and represents an *ellipse* with semiaxes $a = d/2$, $b = \sqrt{(d/2)^2 - m^2}$ and foci at A and B.

A possible landing point of the crashing airplane must be nearer to B than to A and must therefore lie on the right half of this ellipse. Solving Eq. (2·4·5) for x and taking the positive sign in front of the radical, we obtain the locus of possible landing points,

$$x = +a\sqrt{1 - \frac{y^2}{b^2}}$$

The ellipse is a conic section obtained by means of a plane inclined to the axis of the cone and cutting only one of its nappes. Eq. (2·4·2) represents an ellipse when

$$4AB - C^2 > 0 \qquad (2\cdot4\cdot6)$$

c. The Hyperbola

Airports A and B, $2m$ miles apart, belong to two enemy countries, and an espionage agent at A sends information to the authorities at B.

A bomber squadron leaves A in a direction unknown to the foreign agent, who signals immediately the time and type of airplanes to airport B. Ten minutes later, airport B sends in all directions squadrons of fighter airplanes to intercept the bombers. Where will the enemy airplanes meet?

Assuming for simplicity that bombers and fighters have the same speed of, say, v mph, since the fighters left B 10 min ($\frac{1}{6}$ hr) after the bombers left A, the distance traveled by the fighters, say n, must be $c = v/6$ miles shorter than the distance l traveled by the bombers; *i.e.*, calling $P(x,y)$ a possible meeting point,

$$AP - BP = l - n = c$$

But, with the symbols of Fig. 2·7,

$$l = \sqrt{(x+m)^2 + y^2}$$
$$n = \sqrt{(x-m)^2 + y^2}$$

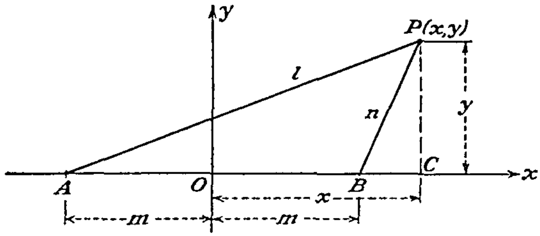


Fig. 2·7.

and hence

$$\sqrt{(x+m)^2 + y^2} - \sqrt{(x-m)^2 + y^2} = c$$

from which, transposing the second radical, squaring, simplifying, and squaring again, we obtain

$$\frac{x^2}{(c/2)^2} - \frac{y^2}{m^2 - (c/2)^2} = 1 \tag{a}$$

From Fig. 2·7 the side l of the triangle PBA is less than the sum of the other two sides,

$$l < n + 2m$$

Therefore

$$m > \frac{l-n}{2} = \frac{c}{2}$$

the denominator of the second fraction of Eq. (a) is positive, and Eq. (a) is of the type

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{2·4·7}$$

representing a *hyperbola*, which crosses the x axis ($y = 0$) at

$$x = \pm\frac{c}{2} = \frac{\pm(l-n)}{2}$$

but does *not* cross the y axis, since for $x = 0$ the values of y given by Eq. (a) are imaginary. The fighters will meet the bombers at a point of this hyperbola nearer to B than to A, that is, at a point on the right branch of the hyperbola, whose equation is

$$x = +a\sqrt{\frac{y^2}{b^2} + 1}$$

$a = c/2$ is called the *transverse semiaxis* of the hyperbola;

$$b = \sqrt{m^2 - \left(\frac{c}{2}\right)^2}$$

is called its *conjugate semiaxis*.

Hyperbola (2·4·7) and the hyperbola

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \qquad (b)$$

are said to be *conjugate* because the transverse semiaxis of one is equal to the conjugate semiaxis of the other, and vice versa (see Fig. 2·8). The intersections of the hyperbolas in Eqs. (2·4·7) and (b) with the coordinate axes are called their *vertices*.

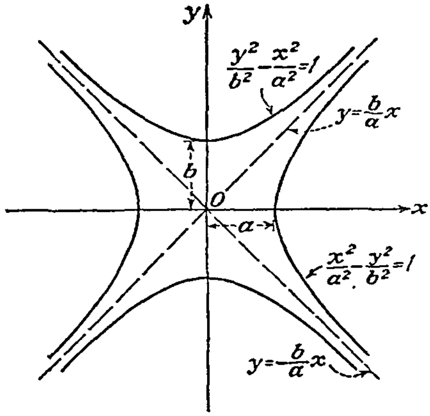


FIG. 2·8.

Deriving the value of y as a function of x from both Eqs. (2·4·7) and (b), we find, respectively,

$$y = \pm b\sqrt{\frac{x^2}{a^2} - 1} \qquad y = \pm b\sqrt{\frac{x^2}{a^2} + 1}$$

When x is so large that unity becomes negligible in comparison with $(x/a)^2$, the ordinates of both hyperbolas approach the common value

$$y = \pm\frac{b}{a}x \tag{2·4·8}$$

Equation (2·4·8) represents two straight lines of slope $\pm b/a$, passing through the origin, called the *asymptotes* of the conjugate hyperbolas. The hyperbolas approach indefinitely their asymptotes without ever touching them. When $a = b$, the asymptotes make 45-deg angles with the x and y axes and are therefore perpendicular to one another. The corresponding hyperbolas are called *equilateral* or *rectangular hyperbolas*. When a rectangular hyperbola of semiaxis a is referred to its asymptotes

as coordinate axes (Fig. 2·9), its equation simplifies to

$$y = \frac{a^2}{2x} \tag{2·4·9}$$

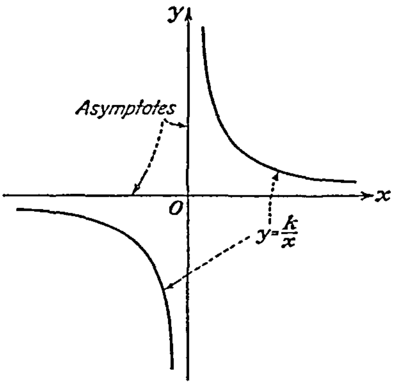


FIG. 2·9.

The hyperbola is a conic section obtained by means of a plane cutting both nappes of the cone. Equation (2·4·2) represents a hyperbola when

$$4AB - C^2 < 0 \tag{2·4·10}$$

d. The Parabola

A bomber, flying horizontally at a speed of 300 mph, drops a bomb on a given target from an altitude of 9000 ft. What path will the bomb follow, and how many seconds before flying over the target must the bomb be released?

If air resistance is neglected, the horizontal speed of the bomb during the fall is equal to 300 mph and its horizontal distance at a time t sec from the instant it is dropped is

$$x = \frac{300 \times 5280}{3600} t = 440t \text{ ft} \qquad (a)$$

Because of the gravitational acceleration g (32.2 ft per sec per sec) the bomb will drop in the same time a vertical distance

$$y = \tfrac{1}{2}gt^2 \qquad (b)$$

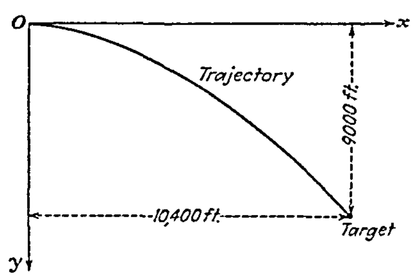


FIG. 2·10.

The path followed by the bomb is obtained by substituting in Eq. (b) the value of t given by Eq. (a), that is, by eliminating t between Eqs. (a) and (b).

$$y = \frac{1}{2} g \left(\frac{x}{440}\right)^2 = 0.0000832x^2 \qquad (c)$$

By Eq. (c), when $y = 9000$ ft, $x = 10{,}400$ ft and, by Eq. (a), $t = 23.7$ sec; hence the bomb must be dropped 23.7 sec before flying over the target. The value of t can also be computed directly from Eq. (b).

$$t = \sqrt{\frac{2y}{g}} = \sqrt{\frac{18{,}000}{32.2}} = 23.7 \text{ sec}$$

Equation (*c*) represents a *quadratic parabola* and is a particular case of the general quadratic equation (2·4·2), which represents a parabola if and only if

$$4AB - C^2 = 0 \tag{2·4·11}$$

A parabola is a conic section obtained by a plane parallel to the side of the cone.

By similarity with Eq. (*c*) the polynomial curves of degree n,

$$y = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

are called *parabolas of degree n*. A first-degree parabola is a straight line.

We have shown in Sec. 2·3 that a straight line can be made to pass through two points [Eq. (2·3·3)]. The coefficients a_2, a_1, and a_0 of a quadratic parabola

$$y = a_2 x^2 + a_1 x + a_0$$

can be chosen so as to have the parabola pass through three points P_1, P_2, P_3; their values are obtained by solving the system of three simultaneous equations in a_2, a_1, and a_0:

$$\begin{aligned} a_2 x_1^2 + a_1 x_1 + a_0 &= y_1 \\ a_2 x_2^2 + a_1 x_2 + a_0 &= y_2 \\ a_2 x_3^2 + a_1 x_3 + a_0 &= y_3 \end{aligned}$$

Similarly, an nth-degree parabola can be made to pass through $(n + 1)$ points. This property of parabolas is very useful in curve-fitting problems.

2·5 Parametric Equations

A real function $y = f(x)$ can always be represented in the x,y plane by means of a curve. Sometimes, as in Eqs. (*a*) and (*b*) of Sec. 2·4 *d*, the coordinates x and y of the point on the curve are given as separate functions of an auxiliary variable t, called a *parameter*.

$$x = x(t) \qquad y = y(t) \tag{2·5·1}$$

Equations (2·5·1) are known as the *parametric equations* of the curve. When the parametric equations can be solved for the parameter, t can be eliminated between them and the Cartesian form $y = y(x)$ of the equation of the curve can be obtained, as was done in Sec. 2·4 *d*.

2·6 Transformation of Coordinates

a. Translation and Rotation

The engineer must never forget that to him mathematics is a useful tool and that his mathematical formulas should always take the simplest possible form.

If one were to measure distances between Stamford, Conn., and New Haven, Conn., it would be unreasonable to set the origin of distances at New York, just because the Post Road begins there. Similarly, in any application of analytic geometry the axes should be originally chosen or later shifted or rotated or both as may appear convenient. When this is done, it is unnecessary to set up all over again the equations of the problem with respect to the new axes. In fact it is seen from Fig. 2·11 that, upon calling x', y' the coordinates of a point referred to new axes parallel to the old axes x, y,

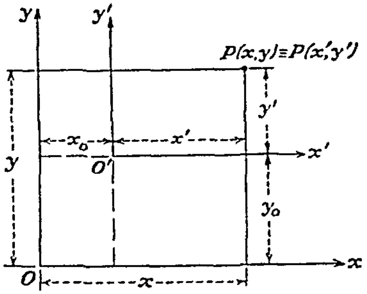


FIG. 2·11.

$$\left.\begin{aligned} x &= x' + x_0 \\ y &= y' + y_0 \end{aligned}\right\} \qquad (2\cdot6\cdot1)$$

where x_0, y_0 are the coordinates of the new origin O' with respect to x, y. The new equations referred to x', y' are obtained by substitution of Eqs. (2·6·1) for x and y in the old equations. For instance, if in the equation of the circle of center x_0, y_0 and radius R

$$(x - x_0)^2 + (y - y_0)^2 = R^2$$

we substitute Eqs. (2·6·1), we obtain the much simpler equation for the same circle referred to the x', y' axes,

$$(x')^2 + (y')^2 = R^2$$

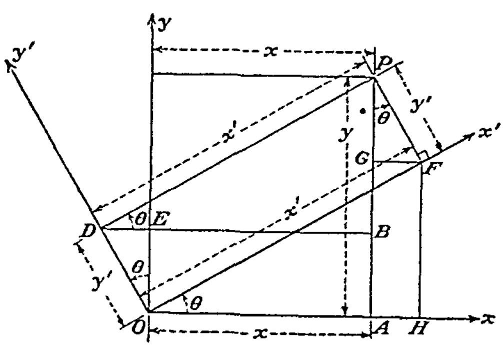


FIG. 2·12.

Very many equations are simplified when referred to a new set of axes x', y' rotated by an angle θ with respect to x, y. From Fig. 2·12 it is

seen that

$$\left.\begin{aligned} x &= OH - HA = OH - FG = x' \cos\theta - y' \sin\theta \\ y &= AP = BP + AB = BP + OE = x' \sin\theta + y' \cos\theta \end{aligned}\right\} \tag{2·6·2}$$

Equations (2·6·2) allow the transformation of equations by rotation. For example, we have seen in Sec. 2·4 *c* that a rectangular hyperbola

$$\frac{y^2}{a^2} - \frac{x^2}{a^2} = 1 \tag{a}$$

can be referred to its orthogonal asymptotes x',y' by a 45-deg rotation. With $\theta = 45°$, letting

$$c = \sin 45° = \cos 45° = \frac{\sqrt{2}}{2}$$

Eqs. (2·6·2) become

$$\begin{aligned} x &= c(x' - y') \\ y &= c(x' + y') \end{aligned}$$

and substitution into Eq. (*a*) gives

$$\frac{c^2(x' + y')^2}{a^2} - \frac{c^2(x' - y')^2}{a^2} = 1$$

from which

$$y' = \frac{a^2}{4c^2}\frac{1}{x'} = \frac{a^2}{2x'}$$

which is identical with Eq. (2·4·8).

When the origin is shifted to x_0,y_0 and at the same time the axes are rotated by θ, the relations between the old and the new coordinates become

$$\left.\begin{aligned} x &= x' \cos\theta - y' \sin\theta + x_0 \\ y &= x' \sin\theta + y' \cos\theta + y_0 \end{aligned}\right\} \tag{2·6·3}$$

b. Polar Coordinates

A point can be located in a plane by measuring its distance r from the origin O and the angle θ between the x axis and the line OP (Fig. 2·13). The quantities r and θ are called the *polar coordinates* of P; r is always considered *positive,* and θ is measured counterclockwise from the positive x axis. From Fig. 2·13 the relationships between the Cartesian and the polar coordinates of a point P are

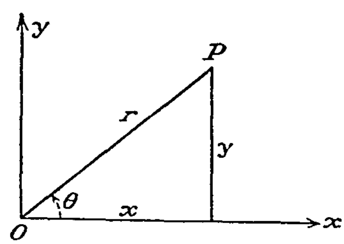


Fig. 2·13.

$$x = r\cos\theta \qquad y = r\sin\theta \tag{2·6·4}$$

Taking the square root of the sum of the squares of Eqs. (2·6·4), we find that

$$r = \sqrt{x^2 + y^2} \tag{2·6·5}$$

while taking the ratio of y to x,

$$\theta = \arctan\left(\frac{y}{x}\right) \tag{2·6·6}$$

Equations (2·6·5) and (2·6·6) give the inverse relationships between polar and Cartesian coordinates. It is important to notice that Eq. (2·6·6) gives two values for θ differing by 180 deg; to choose the correct value of θ it is necessary to refer back to Eq. (2·6·4) and to choose the angle θ for which *both* equations are satisfied. For instance, if $x = -1$, $y = +1$, Eq. (2·6·6) gives

$$\theta = \arctan -\frac{1}{1} = 135° \text{ or } 315°;$$

but since r is always positive, by Eqs. (2·6·4) $\cos\theta$ must be negative and $\sin\theta$ must be positive; hence $\theta = 135°$.

Some equations become simplified when written in polar coordinates; for instance, the equation of a circle $x^2 + y^2 = R^2$ becomes $r = R$; while the equation of the lemniscate $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ becomes $r^2 = a^2 \cos 2\theta$.

2·7 Geometrical Applications of the Calculus

a. Tangent to a Curve

The graph of Fig. 2·14 gives the distance s traveled by a car versus time t. Since the slope of the s curve is measured by the derivative ds/dt (see Sec. 1·8 *a*), the graph indicates that the car traveled with a variable speed $v = ds/dt$. If the speed of the car had remained constant after $t = t_0$, what would have been the distance covered at $t = t_1$?

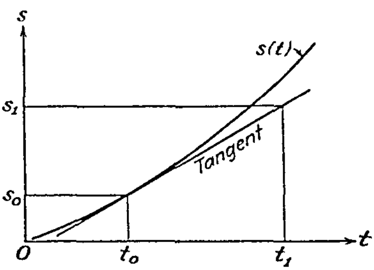


FIG. 2·14.

The s graph for a constant speed is a straight line. If we indicate the slope at $t = t_0$ by $s_0' = ds/dt\Big]_{t=t_0}$, this straight line (the tangent to s at $t = t_0$) must pass through (t_0, s_0) and have a slope equal to s_0'; by Eq. (2·3·1) its equation is

$$s - s_0 = s_0'(t - t_0)$$

and the mileage at $t = t_1$ becomes

$$s_1 = s_0 + s_0'(t_1 - t_0)$$

More in general, the equation of the tangent to a curve $y(x)$ at a point

x_0,y_0, is given by

$$y - y_0 = y'_0(x - x_0) \tag{2·7·1}$$

where $y'_0 = y'(x_0)$.

The tangent to a curve is the limiting position of the chord. In Fig. 2·15 when the chord AB approaches the tangent Y, the two points A,B move toward the point of tangency C. For this reason the tangent is said to have *two points in common with the curve,* both of them located at C. If we indicate by $Y(x)$ the equation of the tangent to the curve $y(x)$ at $x = x_0$, not only does at this point y equal Y, but y' equals Y', since the slope of the tangent is by definition the slope of the curve. When two curves have at a given point the same ordinate and the same slope, we say that they have *a contact of the first order;* hence the tangent has a contact of the first order with the curve.

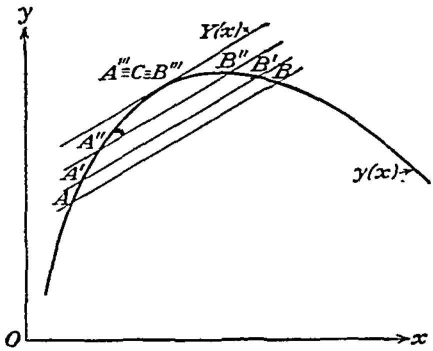


FIG. 2·15.

The graph of Fig. 2·16 illustrates a simple but important property of the derivative of a function. Let the tangent to a curve be oriented along the direction of increasing abscissas. The tangent T at P_1 makes with the positive direction of the x axis, an angle $\theta_1 < 90°$ whose trigo-

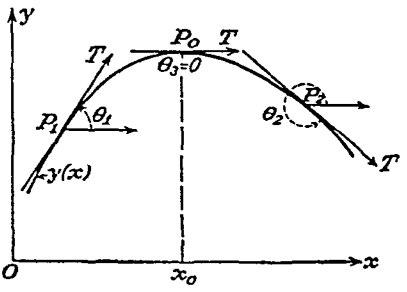


FIG. 2·16.

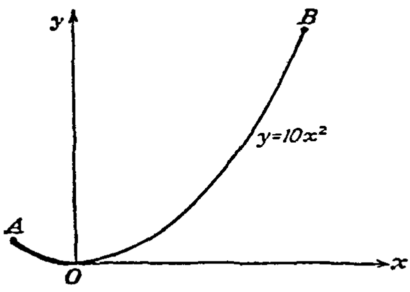


FIG. 2·17.

nometric tangent is positive; therefore at P_1 the slope $y' > 0$. At P_2, $\theta_2 > 270°$, and tan θ_2 is negative; hence $y' < 0$. At P_0, $\theta_3 = 0$, and $y' = 0$. Noticing that $y(x)$ is increasing at P_1, decreasing at P_2, and stationary at P_0, we find the following:

Rule: At points at which a function is increasing, its derivative is positive; at points at which a function is decreasing, its derivative is negative.

b. Curvature

The graph of Fig. 2·17 shows the map of a curve AB built on a highway some years ago, when traffic was relatively slow. Modern cars

entering the sharp curve at high speeds may easily overturn, and a sign must be put up, indicating the maximum safe speed.

We remember from mechanics that the force tending to turn the car over (the so-called "centrifugal force") is equal to

$$F = \frac{W}{g}\frac{v^2}{R} \tag{a}$$

where W is the weight of the car, v its speed, g the acceleration of gravity and R the radius of curvature of the curve. Counteracting the overturning moment Fh of the centrifugal force, where h is the distance of the center of gravity of the car above the road (Fig. 2·18), is the stabilizing moment Wd of the weight of the car, where d is half the axle width. In order to have equilibrium,

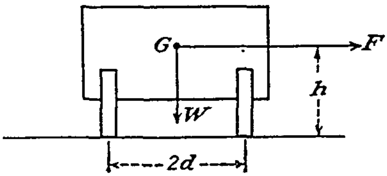


FIG. 2·18.

$$Wd \geq Fh$$

or, by means of Eq. (a)

$$v \leq \sqrt{\frac{gRd}{h}} \tag{b}$$

Equation (b) shows that v is independent of the weight of the car and that, since the ratio d/h does not vary too much from car to car, a single safe speed may be computed.

One of the essential elements in ascertaining the value of v is the radius of curvature R, obtainable directly from the equation of the curve, which in our example is, say, the parabola

$$y = 10x^2 \tag{c}$$

where x and y are measured in miles.

In a curve of *constant* radius R, that is, in a circle, the arc s is measured by

$$s = R\theta$$

where θ is in radians; hence the reciprocal C of the radius R, called the *curvature*, becomes

$$C = \frac{1}{R} = \frac{\theta}{s}$$

Figure 2·19 shows how to evaluate the curvature of a circle by measuring the arc s from a given origin and the angle θ from the positive x axis.

$$s_2 - s_1 = \Delta s = R(\theta_2 - \theta_1) = R\,\Delta\theta$$

from which

$$C = \frac{1}{R} = \frac{\Delta\theta}{\Delta s} \tag{d}$$

When a curve has a variable radius, we *define* its curvature at a point as the limit of Eq. (d) as $\Delta s \to 0$.

$$C = \frac{1}{R} = \lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s} = \frac{d\theta}{ds} \tag{2·7·2}$$

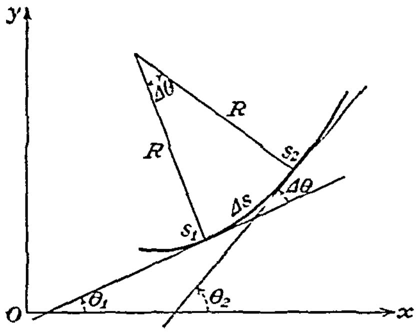


FIG. 2·19.

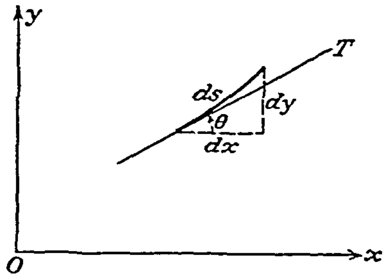


FIG. 2·20.

In order to obtain C directly from the Cartesian equation of the curve, we notice (Fig. 2·20) that the infinitesimal triangle of sides dx, dy, and ds is a right triangle and that its hypotenuse ds, considered straight, has a length

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + (y')^2}\,dx \tag{e}$$

where the plus sign is taken in front of this square root because s increases with x.

Since θ is the angle between the tangent T to the curve and the x axis, $\tan\theta = y'$ and $\theta = \arctan y'$, from which

$$d\theta = \frac{d\theta}{dx}\,dx = \frac{d\theta}{dy'}\frac{dy'}{dx}\,dx = \frac{1}{1 + (y')^2}\,y''\,dx \tag{f}$$

By means of Eqs. (e) and (f), Eq. (2·7·2) becomes

$$C = \frac{1}{R} = \frac{d\theta}{ds} = \frac{y''}{[1 + (y')^2]^{3/2}} \tag{2·7·3}$$

It must be noticed that, since the denominator of Eq. (2·7·3) is always taken positive, the *sign of the curvature is the same as the sign of the second derivative y''*.

The circle, whose radius R is given by Eq. (2·7·3) and whose center is on the inside normal to the curve at a point, is called the *osculating circle* at that point. Since all circles with centers on the normal to a curve and touching it are tangent to the curve, the equation of the osculating circle has the same ordinate, the same slope y', and, by Eq. (2·7·3), the same y'' as the equation of the curve. The osculating circle has at least three points in common with the curve, all of them located at the point of tangency, and the contact between curve and circle is said to be of the second order. A short arc of a curve y is best approximated by an arc of its osculating circle, since the osculating circle is the nearest to the curve among all the tangent circles (*osculare* is the Latin for "to kiss").

We can now compute the radius of the parabolic curve (c) at the point of sharpest curvature $x = 0$ by means of Eq. (2·7·3).

$$y = 10x^2 \qquad y' = 20x \qquad y'(0) = 0 \qquad y'' = 20$$

$$C = \frac{1}{R} = \frac{20}{(1+0)^{3/2}} = 20$$

$$R = 1/20 = 0.05 \text{ mile} = 264 \text{ ft}$$

With $d = 3$ ft and $h = 2$ ft the theoretical maximum safe speed becomes, by Eq. (b),

$$v = \sqrt{32.2 \times 264 \times 3/2} = 113 \text{ ft per sec} = 77 \text{ mph}$$

and, with a coefficient of safety of 2, the maximum speed should be set at about 40 mph.

c. Geometrical Interpretation of the Curvature

It was noticed in Sec. 2·7 *b* that the sign of the curvature is the same as the sign of the second derivative y''. In the curve of Fig. 2·21*a* the

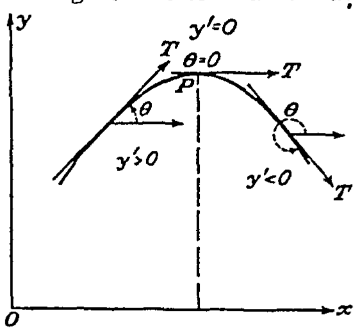


FIG. 2·21*a*.

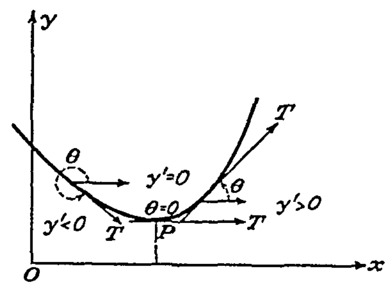


FIG. 2·21*b*.

slope y' is positive to the left of the point P, equals zero at P, and decreases to negative values to the right of P. Hence y' is a *decreasing* function, and *its* derivative y'' by the rule of Sec. 2·7 *a* is *negative* at P. In Fig.

2·21*b*, instead, y' is negative to the left of P, zero at P, and positive to the right of P; hence y' is an *increasing* function, and by the same rule *its* derivative y'' is *positive* at P. But the curve of Fig. 2·21*a* has a concavity pointing toward the *negative* y axis, while the curve of Fig. 2·21*b* has a concavity pointing toward the *positive* y axis; we find, therefore, the following:

Rule: The positive or negative sign of the second derivative of a function $y(x)$, that is, the sign of the curvature of the corresponding curve, indicates whether the concavity of the curve points towards the positive or the negative y axis.

It must be noted that *the sign of the curvature depends upon the orientation of the axes and is not inherent to the curve.*

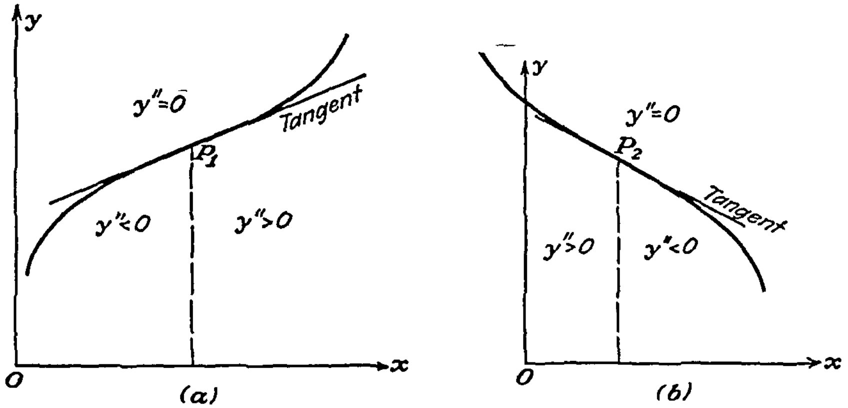


FIG. 2·22.

In Fig. 2·22*a* the curvature is negative to the left and positive to the right of the point P_1; and since y'' is continuous, it must be zero at P_1. For the same reason it is zero at the point P_2 of Fig. 2·22*b*. Points like P_1 and P_2, at which the curvature is zero and the tangent crosses the curve, are called *inflection points.* P_1 is an *increasing* inflection point. P_2 is a *decreasing* inflection point. The points $x = a$ of the two curves

$$y = b + (x - a)^3$$
$$y = b + (a - x)^3$$

are, respectively, increasing and decreasing inflection points, *with a horizontal tangent.* Since any point at which the slope is zero is called a *stationary point,* the points $x = a$ are stationary points but are neither maximum nor minimum points for the curves.

d. Maxima and Minima

The stiffness of a rectangular beam is measured by the moment of inertia I of its cross section with respect to a centroidal axis. I is pro-

portional to the width x and to the cube of the depth y in rectangular sections. For the case of loads parallel to y the constant of proportionality is found to be $\frac{1}{12}$. What is the stiffest beam that can be cut from a circular tree D in. in diameter?

Since, from Fig. 2·23,

$$y = \sqrt{D^2 - x^2}$$

the moment of inertia I becomes

$$I = \frac{xy^3}{12} = \frac{1}{12}x(D^2 - x^2)^{3/2}$$

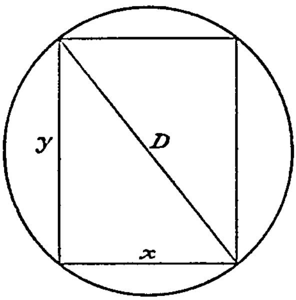


FIG. 2·23.

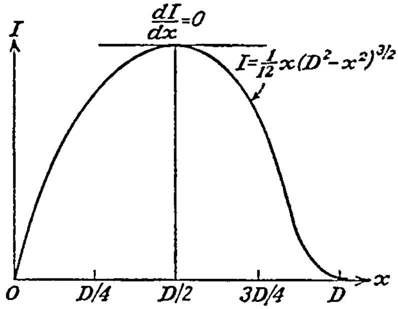


FIG. 2·24.

The graph of I versus x is plotted in Fig. 2·24 and shows that I becomes maximum at $x = D/2$, where the tangent to the graph is horizontal and hence

$$\frac{dI}{dx} = 0$$

To check this we set the derivative of $I(x)$ equal to zero and solve the corresponding equation for x.

$$\tfrac{1}{12}[(D^2 - x^2)^{3/2} - 3x^2(D^2 - x^2)^{1/2}] = \tfrac{1}{12}(D^2 - x^2)^{1/2}(D^2 - 4x^2) = 0$$

This equation has four roots,

$$x = +D \qquad x = -D$$
$$x = +\tfrac{1}{2}D \qquad x = -\tfrac{1}{2}D$$

The first root makes $I = 0$, and the negative roots do not have physical meaning; hence the only root with physical significance is

$$x = \tfrac{1}{2}D$$

The graph of Fig. 2·25 shows that a function y has zero slope at points of minimum value as well. We may therefore state the following fundamental rule:

Rule: At points of maximum or minimum value the first derivative of a function is equal to zero.

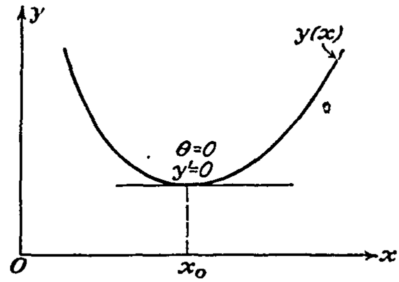


FIG. 2·25.

At points of maximum value the concavity of the curve points down; at points of minimum value the concavity points up. Hence, *with the usual orientation of the x and y axes*, we find the following auxiliary rule:

Rule: At points of maximum value the second derivative of a function is negative; at points of minimum value the second derivative of a function is positive.

As a check of the beam-stiffness problem,

$$\frac{d^2I}{dx^2} = \frac{1}{12}(D^2 - 4x^2)\cdot\frac{1}{2}(-2x)(D^2 - x^2)^{-1/2} + \frac{1}{12}(D^2 - x^2)^{1/2}(-8x)$$
$$= \frac{1}{12}(D^2 - x^2)^{-1/2}x(12x^2 - 9D^2)$$

and

$$\left.\frac{d^2I}{dx^2}\right]_{x=D/2} = \frac{1}{12}\left(D^2 - \frac{D^2}{4}\right)^{-1/2}\frac{D}{2}\left(12\frac{D^2}{4} - 9D^2\right) = -\frac{D^2}{2\sqrt{3}} < 0$$

When at a point the second derivative of a function is zero, together with the first derivative, the behavior of the curve at that point depends on the sign of the higher derivatives, as indicated in the following diagram, which illustrates the various cases of stationary points ($y' = 0$) occurring most frequently in engineering computations.

$$y' = 0\begin{cases} y'' > 0 \ldots\ldots\ldots\ldots & \text{Minimum} \\ y'' < 0 \ldots\ldots\ldots\ldots & \text{Maximum} \\ y'' = 0\begin{cases} y''' > 0 \ldots\ldots & \text{Increasing stationary point} \\ y''' < 0 \ldots\ldots & \text{Decreasing stationary point} \\ y''' = 0\begin{cases} y'''' > 0 \ldots & \text{Minimum} \\ y'''' < 0 \ldots & \text{Maximum} \\ y'''' = 0 \ldots & \text{Stationary point} \end{cases} \end{cases} \end{cases}$$

The maximum and minimum values of a function must not be confused with its largest and smallest values. At maximum and minimum points a function becomes, respectively, larger and smaller than at *neighbouring points*. The function shown in Fig. 2·26 is maximum at B and minimum at E but is larger at E than at B.

The rules for maxima and minima given above are not valid at points at which either the function or its derivatives are discontinuous. Thus points like F, G, or H will never be detected by setting $y' = 0$, for y' is not defined at these points; y is discontinuous at H, and y' is discontinuous at F and G, since the curve has two different slopes there. The rules will also fail at the end points of the interval of definition. In fact, the function of Fig. 2·26 has its largest value at L and its smallest value at A, but its derivative is not equal to zero at either of these two points. It is well to remember, therefore, that the condition $y' = 0$ is neither necessary nor sufficient to locate the smallest and largest values of a function in a given interval.

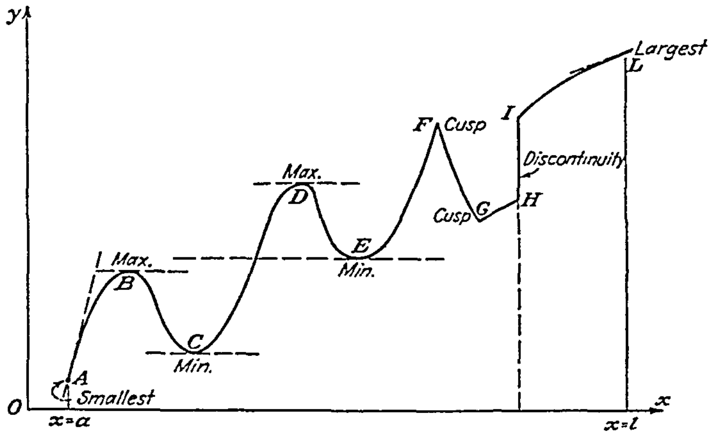


FIG. 2·26.

Problems

1. Compute the distance between each of the following pair of points and the slope of the line connecting the first with the second:

(a) (0, 1), (5, 7) (b) (−2, 3), (4, 2)
(c) (7, 6), (6, 7) (d) (−3, −2), (−4, −1)
(e) (−3, 8), (10, 4) (f) (2.4, −5.1), (7.2, 3.1)
(g) (−3.1, −1.4), (−4.2, −1.2) (h) (0, 4.3), (0, 3.3)
(i) (0, −1), (−1, 0) (j) (2.25, 8.64), (−2.34, 8.22)

2. Write the equation of the following lines:

(a) through (0, 2) with slope $m = 0.5$
(b) through (−1, −4) with slope $m = -\frac{2}{3}$
(c) through (0, 0) with slope $m = \infty$
(d) through (−4, −3.2) with slope $m = 0$

3. Write the equation of the following lines; compute their slope, when not given, and their x and y intercepts:

(a) through (2, 4), (6, 1)
(b) through (−4, 1), (−7, −2)
(c) through (0, −1), (2, −4)
(d) through (0, 0), (0, 2)
(e) through (3, 6) with slope $m = 0.75$
(f) through (2, 2) with slope $m = 1$

4. Compute the slope and the x and y intercepts of the following lines:

(a) $3x + 2y - 4 = 0$ (b) $2x - 2y - 3 = 0$
(c) $4x + y = 0$ (d) $-6x + 7y + 14 = 0$
(e) $24.2x - 12.3y + 9.7 = 0$ (f) $-13.25x + 34.12y - 64.25 = 0$

5. Compute the intersection of the following lines:

(a) $x + 2y - 2 = 0$ $3x + 2y + 4 = 0$
(b) $-2x - 3y + 7 = 0$ $2x - 4y + 14 = 0$
(c) $x + y = 0$ $x - y = 0$
(d) $x - 4 = 0$ $y - 2 = 0$

6. Write the equation of the 2 lines through the given point, respectively parallel and perpendicular to the following lines:

(a) (0, 1) $2x + 4y - 7 = 0$ (b) (−2, −3) $x - 4y + 3 = 0$
(c) (2.1, −4.7) $2.4x - 7.2y - 6.4 = 0$
(d) (3.2, 7.9) $-4.1x + 9.2y + 10.2 = 0$
(e) (0, 0) $x = 24$ (f) (1,0) $y = 13.1$
(g) (0, 1.14) $-x + y = 0$ (h) (1.1, −2.2) $x + y = 0$

7. Compute to the nearest degree the angle between the pairs of lines of Prob. 5.
8. Compute the distance between the given points and lines of Prob. 6.
9. Prove that the segment joining the mid-points of two sides of a triangle is parallel to the third side and equal to half of it in length. [Take the vertices of the triangle to be (0, 0), (a, 0), (b,c).]
10. Prove that the intersections of the diagonals of a trapezoid and the mid-points of the parallel sides lie on a straight line.
11. A line through the point (3, 4) makes an angle of 135 deg with the x axis. Compute the area of the triangle it forms with the x,y axes.
12. (a) Find the distance between the 2 parallel lines

$$y + 3x - 7 = 0 \qquad y + 3x - 12 = 0$$

(b) Derive a general expression for the distance between 2 parallel lines

$$Ax + By + C = 0$$
$$Ax + By + D = 0$$

13. If (4, 3) and (−2, 7) are the vertices of an equilateral triangle, find the third vertex (2 answers).
14. A square $ABCD$ moves so that its vertices A and C always remain on the y and the x axis, respectively. Show that the vertices B and D also move on two fixed mutually perpendicular straight lines.

15. Find the area of the triangle whose vertices have the following coordinates: (2, 3) (−4, 7), (0, 6).

16. The rise and fall of a cam are plotted versus its angular displacement in Fig. 2·27. What is the linear speed of the follower in inches per second if the cam is rotating at 60 rpm?

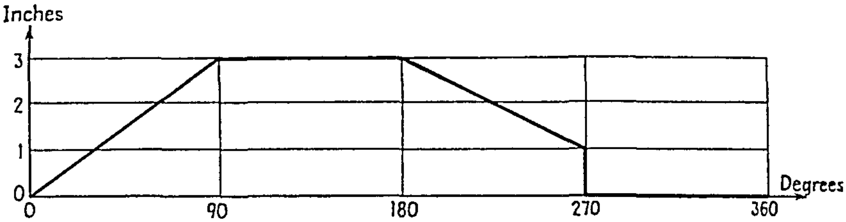


FIG. 2·27.

17. An automobile starts at noon and travels at a uniform speed of 40 mph until 2 P.M. It then develops engine trouble and remains at a gasoline station until 3 P.M., when it starts traveling again at a uniform speed of 50 mph. This speed is maintained until a total distance of 100 miles is covered. A second automobile starts for the same destination at 2 P.M. at a speed of 20 mph, which is increased linearly at the rate of 40 mph every hour until it reaches the same destination. Which automobile gets there first and how much sooner? Solve graphically and analytically.

18. Write the equations of the following circles:

(a) center at (2, 1), radius $r = 2$
(b) center at (−2.4, 1.2), radius $r = 4.8$
(c) center at (0, 1), through (2, 4)
(d) center at (1.1, 4.2), through (−4.1, −2.2)
(e) through (1, 2), (0, 3), (−3, −1)
(f) through (0, 0), (4.2, 2.4), (1.2, 5.4)

19. Locate the center and compute the radius of the following circles:

(a) $x^2 + y^2 + 4x - 2y = 0$
(b) $2x^2 + 2y^2 - 3y - 1 = 0$
(c) $x^2 + y^2 + 2x + 3y - 2 = 0$
(d) $3.1x^2 + 3.1y^2 - 7.4x + 1.2y - 1.6 = 0$

20. Find the locus of a point that moves so that the sum of the squares of its distances from the 4 sides of a square remains constant.

21. Two lines L_1 and L_2, passing through 2 fixed points A and B, respectively, always intersect at right angles. Prove that the locus of their intersections is a circle of radius half the distance A to B. Take the 2 points A and B to be $(-r, 0)$ and $(r, 0)$.

22. Find the central angle subtended in the circle

$$x^2 + y^2 - 4x - 2y = 0$$

by the chord $x + y = 0$.

23. Write the equations of the following ellipses:

(a) major semiaxis 3, minor semiaxis 2, center at the origin
(b) vertices (−4, 0), (4, 0), (0, −3), (0, 3)

(*c*) sum of the distances of any point on the ellipse from the foci $= 10$; major semiaxis equal 3 times minor semiaxis. Take the foci to be the points $(-c, 0)$, $(c, 0)$

24. Write the equations of the following hyperbolas:

(*a*) foci $(\pm 2, 0)$, vertices $(\pm 1, 0)$
(*b*) vertices at $(\pm 2, 0)$, difference of the distances from the foci 5
(*c*) center at the origin, transverse semiaxis 3, conjugate semiaxis 4
(*d*) hyperbolas conjugate of hyperbolas (*a*), (*b*), and (*c*)

25. What are the equations of the asymptotes of the following hyperbolas?

(*a*) $4x^2 - 6y^2 - 9 = 0$ (*b*) $-2x^2 + 3.1y^2 - 4 = 0$
(*c*) $xy = 24$ (*d*) $3xy + 12 = 0$

26. Write the equations of the following quadratic parabolas:

(*a*) tangent to the x axis at $x = 2$, passing through $(1, 2)$
(*b*) tangent to the x axis at the origin, passing through $(1, -2)$
(*c*) passing through $(0, 1)$, $(2, 3)$, $(-2, 4)$
(*d*) passing through $(0, 0)$, $(-0.8, 2.1)$, $(2.1, 3.5)$

27. Reduce the equation $Ax^2 + By + Cx + D = 0$ to the standard form of the quadratic parabola $y = cx^2$ by a suitable translation of the axes.

28. An equilateral triangle with one vertex at the origin has the other two vertices on the parabola $y = 4x^2$. What is the length of its sides?

29. A parabola can be defined as the locus of a point whose distance from a fixed line is equal to its distance from a fixed point, called the *focus*. From this definition derive the standard form of the equation of the parabola. [Take the line to be $y = -c$ and the point to be $(0, c)$.]

30. Parabolic mirrors are used as reflectors because a source of light placed at the focus (see Prob. 29) of the parabola will reflect parallel rays of light. Prove this statement. *Hint:* The angle of incidence equals the angle of reflection.

31. Determine whether the following conic sections are circles, ellipses, hyperbolas, or parabolas:

(*a*) $2x^2 + 2y^2 - 3xy + 10 = 0$ (*b*) $x^2 + 2y^2 - 4x + y - 12 = 0$
(*c*) $x^2 + y^2 + x + y = 0$ (*d*) $-x^2 + y^2 - 12xy + 24y + 3 = 0$
(*e*) $x^2 + y^2 + 2xy - 3x + 12 = 0$ (*f*) $2x^2 - y^2 - 3x - 2y + 4 = 0$

32. Find the loci defined by the following parametric equations:

(*a*) $x = 2 \sin \alpha$ $y = 2 \cos \alpha$
(*b*) $x = 2 \sin \alpha$ $y = 3 \cos \alpha$
(*c*) $x = 3 \cosh \alpha$ $y = 3 \sinh \alpha$
(*d*) $x = 2t + 3$ $y = t^2 - 4$
(*e*) $x = 3\sqrt{t}$ $y = \sin 2t$
(*f*) $x = 2e^t$ $y = 3e^{-t}$

33. Write the equations of the following curves referred to a new set of axes parallel to x,y with origin at the point indicated:

(a) $x^2 + y^2 - 2x + 2y - 2 = 0$ (1, −1)
(b) $xy - 3x - 2y + 2 = 0$ (2, 3)
(c) $x + y - 6 = 0$ (4, 2)
(d) $2x^2 - 4y^2 + 16y + 4x - 23 = 0$ (−1, 2)

34. Write the equations of the following curves referred to a new set of axes rotated by the given angle with respect to the x,y axes, but with the same origin.

(a) $x^2 + y^2 - 4 = 0$ $\theta = \theta$
(b) $2x^2 + y^2 + xy - 2 = 0$ $\theta = 22.5°$
(c) $xy = \frac{1}{2}$ $\theta = 45°$
(d) $x - y = 2$ $\theta = 45°$

35. Write the equation of the ellipse whose major semiaxis equals 2 and lies along a line making a 30-deg angle with the x axis, whose minor semiaxis equals 1, and whose center is at (1, −3). *Hint:* Start with the equation $(x^2/a^2) + (y^2/b^2) = 1$, rotate the axes by 30 deg, and shift the origin to (−1, 3).

36. Show that if the curve

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is referred to a new set of axes rotated by an angle θ, such that $\tan\theta = B/(A - C)$, the term xy disappears.

37. By rotation and translation, reduce

$$x^2 + y^2 + xy - 3x - 2y - 7 = 0$$

to the form $(x/a)^2 + (y/b)^2 = 1$.

38. Reduce the equation

$$3x^2 - 3y^2 - 4x + 4y - 9 = 0$$

to the form $x^2/a^2 - y^2/b^2 = 1$ by a suitable shift of the origin.

39. Reduce the equation

$$4x^2 + 7y^2 - 3x + 4y - 3 = 0$$

to the form $(x^2/a^2) + (y^2/b^2) = 1$ *by a suitable shift of the origin.*

40. Write the equation of the asymptotes of the hyperbola

$$xy - 3y - 6 = 0$$

41. What is the center of gravity of a system of particles of equal mass located at (0, 1), (4, 7), (6, 3), (−2, 4), (−7, −6). Solve graphically and analytically. *Hint:* Coordinates of the centroid of particles m_i,

$$\bar{x} = \frac{\Sigma m_i x_i}{\Sigma m_i} \qquad \bar{y} = \frac{\Sigma m_i y_i}{\Sigma m_i}$$

42. Express in polar coordinates the following equations:

(a) $2x + y - 7 = 0$
(b) $x^2 + y^2 - x = 0$
(c) $2x^2 + 4y^2 - 9 = 0$
(d) $2x^2 - 4y^2 - 16 = 0$
(e) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
(f) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
(g) $xy = c$
(h) $y - a^2x^2 = 0$

43. Transform the double integral

$$\int_0^\infty \int_0^\infty f(x,y)\,dx\,dy$$

into an equivalent integral by means of polar coordinates.

44. By transformation to polar coordinates, evaluate

$$I = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Hint:

$$I^2 = \int_0^\infty e^{-x^2}\,dx \int_0^\infty e^{-y^2}\,dy$$
$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

45. Write the equation of the tangent to the following curves at the given point:

(a) $y = x^2 + 2x - 4 \qquad x = 1$

(b) $y = \sin 2x \qquad x = \frac{\pi}{4}$

(c) $y = 3\cos 3x \qquad x = 60°$

(d) $y = 4.2e^{0.5x} \qquad x = 1$

46. Write the equation of the tangent to the following curves *from* the given point:

(a) $y = x^3 + 24 \qquad (0, 0)$

(b) $y = e^{-x} \qquad (0, 0)$

(c) $y = \sin 2x \qquad (0, 1)$

(d) $x^2 + y^2 = 4 \qquad (4, 0)$ (two answers)

47. Determine the equation of the family of lines all of which are tangent to the circle $x^2 + y^2 = 1$.

48. Find the equation of the conic that passes through the 5 points (1, 0), (3, 1), $(7, \sqrt{3})$, (9, 2), (0, 0). *Hint:* The general equation of the conic is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

49. Determine whether the following curves are increasing, decreasing, or stationary at the given point:

(a) $y = 2.1x^3 - 1.2x^2 + 4.7 \qquad x = 1.2$

(b) $y = 3.2\cos 2.4x \qquad x = 0.41$

(c) $y = 7e^{0.61x} \qquad x = 0.12$

(d) $y = 2.1e^x - 4.2e^{-x} \qquad x = 0.25$

(e) $y = 3\sqrt{1 - \frac{x^2}{4}} \qquad x = 0, x = +1, x = -1$

(f) $xy + 12 = 0 \qquad x = 2$

50. Determine the value of the curvature of the curves of Prob. 49 at the given points.

51. Determine the curvature at the end of the semiaxes of the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

(see Prob. 53).

52. Determine the curvature of the hyperbola

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1 \quad \text{at} \quad x = \pm a$$

53. Prove that the curvature of a curve may be expressed by

$$C = \frac{-x''}{[1 + (x')^2]^{3/2}}$$

where $x' = dx/dy$ and $x'' = d^2x/dy^2$.

54. Find the curvature of the curve $x = \sec t$, $y = \tan t$ at the point $(-1, 0)$. *Hint:* Evaluate the derivatives of y with respect to x as a ratio of differentials.

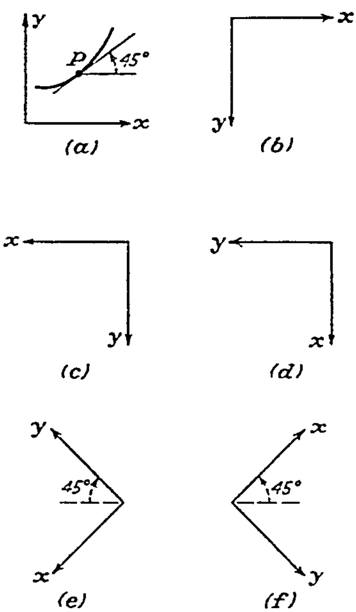


FIG. 2·28.

55. Determine the sign of the curvature of the point P of the curve $y = y(x)$ of (Fig. 2·28, when the axes are oriented as shown.

56. Examine the following curves for maximum, minimum, and inflection points.

(a) $y = x^2 - 4x + 2$
(b) $y = x^3 - 7x + 4$
(c) $y = 3x^3 + 2x^2 - 4x + 4$
(d) $y = 2x^4 - x^2 + 1$
(e) $y = x^3 - 1$
(f) $y = (x - 1)^3$

57. Find the dimensions of the rectangle of maximum area that can be inscribed

(a) in a circle of radius a
(b) in a semicircle of radius a
(c) in an ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

58. Find the dimensions of the rectangular gutter of maximum carrying capacity that can be cut out of a long, thin piece of sheet metal 10 in. wide.

59. Why is a portion of a semicubical parabola used as the initial part of a railroad curve when the track deviates from a straight course? (A semicubical parabola is a curve of the type $y = cx^{3/2}$.)

60. If in the van der Waals equation

$$V^3 - \left(b + \frac{RT}{P}\right)V^2 + \frac{aV}{P} - \frac{ab}{P} = 0$$

T is the critical temperature T_c, both the first and the second derivative of P with respect to V are zero. Show that in this case

$$a = 3V_c^2P_c$$
$$b = \frac{V_c}{3}$$
$$R = \frac{8P_cV_c}{3T_c}$$

61. What are the angles of an isosceles triangle for which the sums of their sines are a maximum?

62. What is the largest right-circular cylinder that can be inscribed in a sphere of radius a? *Hint:* Consider $r = \sqrt{x^2 + y^2}$ as the independent variable.

63. A company is planning the construction of a new office building. The cost of the building varies as the square of its height, and each floor can be rented at $1000 a month. How many stories should the building have in order that the owner make a maximum profit after 10 years of occupancy? The cost of a 10-story building is $100,000.

64. A rectangular piece of galvanized iron, 2 by 4 ft, has a square cut out at each corner, such that the box formed by the sheet has a maximum volume. What are the sides of each square and the resulting volume?

65. A bullet is fired vertically with an initial velocity of v_0 ft per sec. Neglecting air resistance, how high will the bullet go?

66. A company is making saucepans with a circular cross section. What should be the ratio of height to diameter in order that the pans will have maximum capacity for a given amount of material used in their construction?

67. What is the largest rectangular area that can be fenced off with 1000 ft of fence, if one side of the rectangle is bounded by a river?

68. A fire ladder must pass over a garage 8 ft high and 27 ft wide to reach the side of a burning house. What is the minimum length of ladder than can be used?

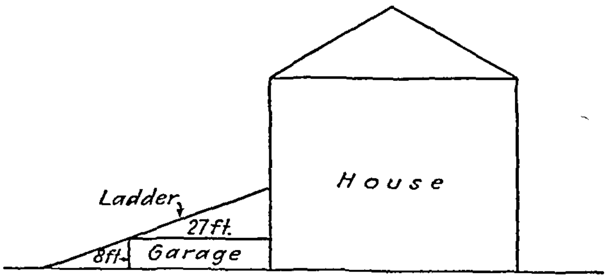


FIG. 2·29.

69. The price of transporting farm machinery varies directly as the square of the distance from the factory to the consumer. With the present location of the factory the number of machines sold varies inversely as the cube of the distance from the producer to the farm. If the unit cost of a machine is $250 and the cost of transporting one piece of equipment 100 miles is $30, at what distance from the factory is the company doing the best business?

70. A trough with parabolic cross section $y = 4x^2$ is filled with water to a depth H. At what distance y from the bottom of the trough is the force per unit of depth exerted against the ends of the trough a maximum?

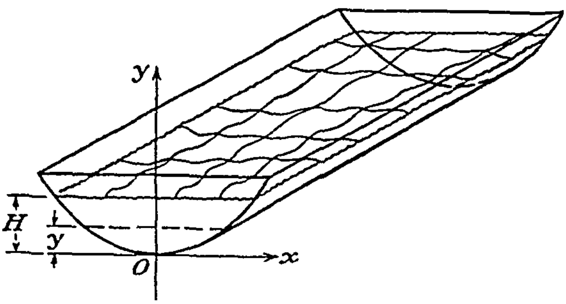


FIG. 2·30.

71. The deflection w of a certain simply supported beam of length 40 ft is given by

$$1000w = x^3 - 30x^2 - 400x$$

where x is the distance measured from the left support A. At what distance x_0 from A does the maximum deflection occur?

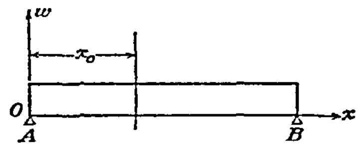


FIG. 2·31.

72. A pressure vessel is to be made in the shape of a right-circular cylinder with a flat base and a hemispherical top. If its total volume is to be 2000 cu ft, what must the dimensions of the vessel be in order that a minimum amount of material will be used in its construction?

73. A small jewelry box is to be made of expensive wood with a sterling-silver top. If silver costs five times as much as the wood used, what should be the dimensions of a rectangular box of 32 cu in. capacity for its cost to be a minimum? Assume the box has a square transverse cross section.

74. What will be the ratio of height h to radius of the base r in a cylinder, of given capacity V, whose total surface is a minimum?

75. A plate is subjected to normal and shear stresses as indicated in Fig. 2·32. Calculate the angle α of a section on which the stress σ will be a maximum. What is the corresponding value of τ?

Hint:

$$\sigma = \sigma_x \cos^2 \alpha + \sigma_y \sin^2 \alpha + 2\tau_{xy} \sin \alpha \cos \alpha$$
$$\tau = (\sigma_y - \sigma_x) \sin \alpha \cos \alpha + \tau_{xy}(\cos^2 \alpha - \sin^2 \alpha)$$

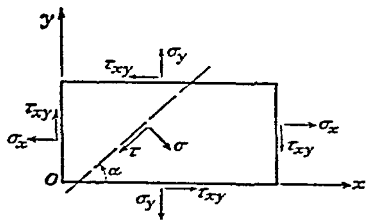


FIG. 2·32.

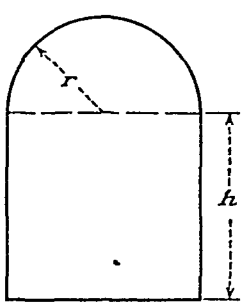


FIG. 2·33.

76. An architect wishes to design a window for a church in the shape of a rectangle of height h surmounted by a semicircle of radius r. What must be the ratio of r to h for the window to admit a maximum of light, for a given amount of molding surrounding the window?

77. A field gun has a muzzle velocity of v ft per sec. Neglecting air resistance, at what angle of elevation θ should the gun be fired to attain maximum range?

78. A mine tunnel is to be built from A to B (see Fig. 2·34) through an upper layer of shale and a lower layer of hard packed earth. If the cost of excavating 1 cu yd of shale is 4 times the cost of excavating 1 cu yd of earth, through what point C should the tunnel pass to reduce the excavating cost to a minimum? (Solve the equation for x by trial and error.)

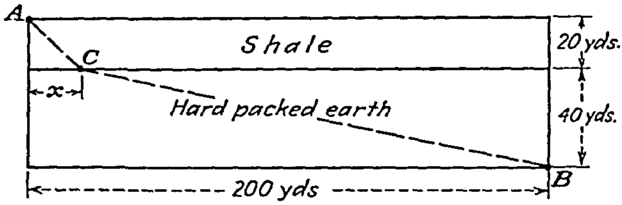


FIG. 2·34.

79. A man is drowning at sea 50 ft from the shore. Another man runs to his help from a point 25 ft from the shore and 100 ft from the drowning man along the shore. The would-be rescuer can run at 10 mph and swim at 2 mph. What path should he follow to reach the drowning man in the shortest time? (Solve the equation for x by trial and error.)

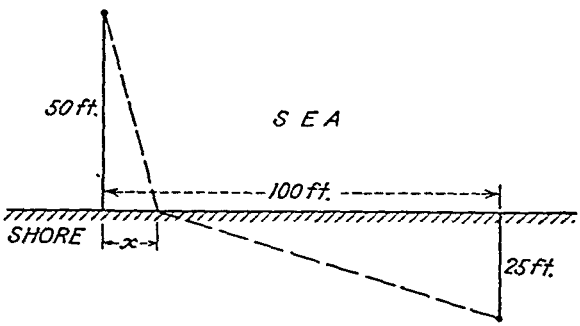


FIG. 2·35.

80. Two towns A and B are to get water from a river. A is 6 miles and B 18 miles back from the right bank of the river, while A is 10 miles downstream with respect to B on a straight stretch of river. Where should the pumping station be located to use a minimum amount of pipe?

81. The intensity of light varies inversely as the square of the distance from its source. Two lights are 1 mile apart, and one is twice as strong as the other. At what point of the line connecting the lights is illumination a minimum?

82. A manufacturing company wishes to build a heat exchanger. If the cost of 10-ft-long tubes is \$10 per inch of diameter and the company needs 250 sq ft of area for the exchanger, what diameter tubes would it be most economical to use?

83. At what velocity v will an atomic particle have its maximum kinetic energy if its "rest" mass is m_0? The mass of the particle in motion is

$$m = \frac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

where c is the velocity of light.

84. The hourly cost of driving a steamer through water is proportional to the cube of its speed. Find the most economical speed at which to drive the steamer a given distance against a current of a knots.

85. The velocity of light in a refracting medium is inversely proportional to the index of refraction of the medium. Prove that the path taken by a light ray from A to B is such that the ratio of the sines of the angles of incidence i and refraction r equals n_2/n_1.

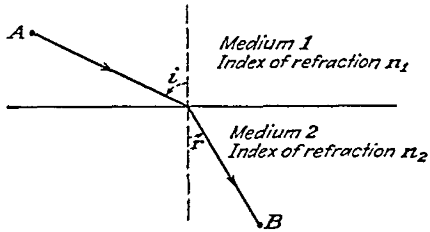


FIG. 2·36.

86. A concern wishes to build a series of identical apartment houses. They are furnished the following cost sheet:

Number of Stories	Cost
1	1,000
2	2,500
3	4,500
4	7,000
5	10,000
.	.
.	.
.	.

A plot of land for each house costs \$10,000. The company has \$100,000 available. How many lots should they buy, and how many floors should each apartment have?

87. Inscribe in a sphere of radius R a right circular cone whose total surface is a maximum.

88. A simply supported beam of length L and flexural rigidity EI is loaded by a linearly varying load $q = q_0(x/L)$ lb per ft. Remembering that the deflection function y satisfies the equation

$$EI\,\frac{d^4y}{dx^4} = q$$

and the conditions $y(0) = y(L) = 0$, $y''(0) = y''(L) = 0$, determine the abscissa of maximum deflection. (The abscissa x is measured from one end of the beam.)

89. The sum of length and girth of packages for parcel-post shipment is limited to 72 in. What are the largest packages that can be mailed

(a) in the shape of a rectangular parallelepiped of square base
(b) in the shape of a circular cylinder

90. Two ships have their straight-line courses intersecting at an angle of 30 deg. If their distances from the point of intersection at a certain time are a and b, respectively, find the least distance between the ships if their constant velocities are v_1 and v_2, respectively.

91. A generator with an internal impedance Z_g of R_g ohms (resistive) and X_g ohms (reactive) feeds a load of impedance Z_L, R_L (resistive) and X_L (reactive). What must the values of R_L and X_L be such that the generator delivers maximum power to the load? *Hint:* Power $= I^2R$. Assume a schematic circuit as shown in Fig. 2·37. $Z_g = R_g + iX_g$; $Z_L = R_L + iX_L$.

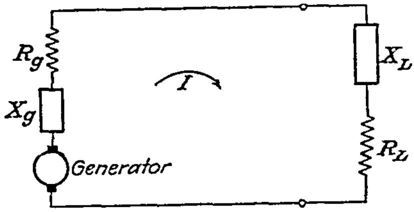


FIG. 2·37.

92. A vertical tank is 20 ft in diameter and has a 4-in.-diameter circular hole in its bottom. It is found that, when water is 49 ft. high in the tank, the tank discharges it at the rate of 1.00 cu ft per sec through the hole. What should be the diameter of the hole to discharge a 50-ft head in exactly 4 hr? *Hint:* Assume the rate of discharge to be proportional to the square root of the head of fluid above the opening and to the area of the small hole.

93. A point moves on a path whose parametric equations are

$$x = t^3 - 1 \qquad y = 10 - t^3$$

where t is time in seconds. At what point of the path is the velocity V of the point a minimum? *Hint:*

$$V = \sqrt{(dx/dt)^2 + (dy/dt)^2}.$$

94. A point moves on a path whose parametric equations are

$$x = 2t^2 - 6 \qquad y = 3t^2 - 4$$

where t denotes the time in seconds. At what points of the arc described between $t = 0$ and $t = 2$ sec is the speed of the point smallest and largest, respectively?

95. A portion of a bus bar has the shape of a semicircle of inner and outer radii a and A, respectively. If the bar's width is D units and its resistivity ρ, compute the resistance R of the bus bar. *Hint:* $1/R = G = A/\rho L$, where G is the conductance, A is the cross-sectional area; L is the length, and ρ is the resistivity.

96. A heavy-duty punch and shear machine requires 250 kw at full load and operates at this load with 70 per cent efficiency. Its efficiency decreases linearly with overload and at 20 per cent overload is reduced to 50 per cent. If the machine produces 1000 items an hour at a profit of 10 cents apiece running full load, and if power costs 6 cents per kilowatt-hour, at what percentage overload should the machine be run in order to make a maximum profit?

97. A rotary filter is 5 ft in diameter and 6 in. wide. The "cake" filters out at a rate inversely proportional to the thickness of cake already filtered. After running the filter for 1 hr, 1 in. of cake has been deposited. If the cake can be sold at a profit of 1 cent per cubic inch and it costs \$5 an hour to run the filter, what is the optimum length of time to filter each batch?

98. In order to promote the sale of new cars that sell at \$1500, the manufacturing company offers its salesmen a bonus of \$100 for the first, \$110 for the second, \$120 for the third, etc., car sold in excess of the first 50 cars sold per month. The regular commission is 10 per cent for each car. How many cars must each salesman sell a month in order that the company may make a maximum gross profit?

99. A light is to be placed on a wall so as to illuminate a desk S ft from the wall. Assuming that the illumination varies inversely as the square of the distance and directly as the sine of the angle of inclination of the rays on the desk, how high above the desk should the light be placed?

CHAPTER III

THE NUMERICAL SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

3·1 The Linear Equation

A bar of a bridge truss is subjected to a maximum tensile force of 300,000 lb. If the allowable unit stress of the steel used in manufacturing the truss is 15,000 psi, what must be the cross section of the bar?

This is probably the simplest design problem an engineer can be confronted with. Calling x the unknown cross-sectional area of the bar and equating the external force to the sum of the internal stresses, we find that x must satisfy the first-degree equation

$$15{,}000x = 300{,}000$$

from which

$$x = 20 \text{ sq in.}$$

Similarly, the most general linear equation with real coefficients in one unknown,

$$ax + b = 0 \tag{3·1·1}$$

has always one real root

$$x = -\frac{b}{a} \tag{3·1·2}$$

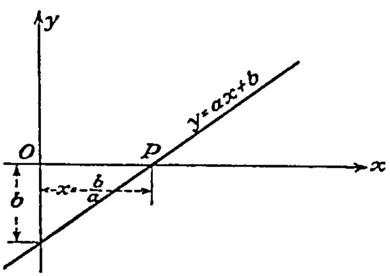


FIG. 3·1.

Calling y the left-hand member of Eq. (3·1·1),

$$y = ax + b \tag{3·1·3}$$

and considering Eq. (3·1·3) as the equation of a straight line in Cartesian coordinates, we see that the determination of the root (3·1·2) is geometrically equivalent to the location of the intersection of the line with the x axis (Fig. 3·1).

3·2 The Quadratic Equation

A rocket is launched vertically at an initial speed of 4150 ft per sec. In how many seconds will it reach an altitude of 50 miles?

Neglecting air resistance and measuring time from the moment of the launching, at a time t the rocket will have moved upward 4150 t ft

owing to its initial speed and $\frac{1}{2}gt^2$ ft downward owing to gravitational acceleration ($g = 32.2$ ft per sec per sec); its altitude s is therefore

$$s = 4150t - 16.1t^2$$

and an altitude of 50 miles, *i.e.*, of 264,000 ft, will be reached at a time defined by the quadratic equation

$$264{,}000 = 4150t - 16.1t^2$$

or

$$16.1t^2 - 4150t + 264{,}000 = 0 \tag{a}$$

It is well known from algebra that a quadratic equation

$$ax^2 + bx + c = 0 \tag{3·2·1}$$

has two roots given by

$$\left.\begin{aligned} x_1 &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} + \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} \\ x_2 &= \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} - \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} \end{aligned}\right\} \tag{3·2·2}$$

Applying Eq. (3·2·2) to Eq. (a), we find that an altitude of 50 miles will be reached when

$$t = \frac{4150 \pm \sqrt{4150^2 - 4 \times 16.1 \times 264{,}000}}{2 \times 16.1} = \begin{cases} 143 \text{ sec} \\ 114 \text{ sec} \end{cases}$$

The question asked by the problem has two mathematical answers, both of which, in the present case, have physical significance. The rocket will reach an altitude of 50 miles at $t = 114$ sec while moving up and at $t = 143$ sec while coming down. In other cases only one root may have physical significance, and it is often up to the applied mathematician to find which root must be discarded because of the conditions of the problem.

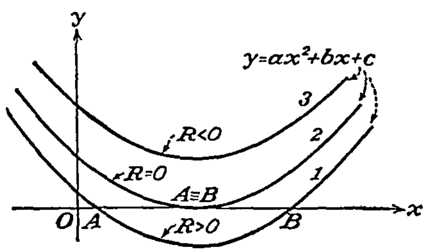


FIG. 3·2.

It will be noticed that Eqs. (3·2·2) give two real, separate roots for Eq. (a), because the quantity under the square root

$$R = b^2 - 4ac \tag{3·2·3}$$

is in this case greater than zero. When R is equal to zero, the two roots are real but coincident; when R is less than zero, the two roots become conjugate complex numbers. This is illustrated in Fig. 3·2, where the graph of the function

$$y = ax^2 + bx + c \tag{3·2·4}$$

is plotted versus x. This quadratic parabola may cross the x axis at two separate points A,B ($R > 0$), may be tangent to the axis ($R = 0$), or may be entirely above or entirely below the axis ($R < 0$).

Dividing Eq. (3·2·1) by a

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

and writing the quadratic equation in the factored form

$$(x - x_1)(x - x_2) = 0$$

we find that

$$\frac{b}{a} = -(x_1 + x_2) \qquad \frac{c}{a} = x_1x_2 \tag{3·2·5}$$

3·3 The Biquadratic Equation

When an inextensible cable of length L ft is suspended from two points in a horizontal plane a ft apart and loaded uniformly along the horizontal, its sag f can be approximately determined by means of the equation

$$L = a\left[1 + \frac{8}{3}\left(\frac{f}{a}\right)^2 - \frac{32}{5}\left(\frac{f}{a}\right)^4\right] \tag{a}$$

(see Sec. 5·8). This equation is derived under the assumption that the weight of the cable is negligible in comparison with the load and that the sag is small in comparison with a, so that powers of f/a higher than the fourth can be neglected.[1] If an inextensible cable of length $L = 100$ ft is suspended from two points 99 ft apart, what will its sag be?

Calling x the ratio f/a, Eq. (a) becomes

$$\tfrac{32}{5}x^4 - \tfrac{8}{3}x^2 + \tfrac{100}{99} - 1 = 0$$

or

$$6.40x^4 - 2.67x^2 + 0.0101 = 0 \tag{b}$$

An equation of the fourth degree, lacking the third- and first-power terms, is called a *biquadratic equation* and is solved by letting $x^2 = y$ and by solving the quadratic equation thus obtained,

$$6.40y^2 - 2.67y + 0.0101 = 0 \tag{c}$$

By means of the two roots of Eq. (c),

$$y_1 = 0.421 \qquad y_2 = 0.00374$$

the four roots of Eq. (b) become

$$\begin{aligned} x_1 &= +\sqrt{0.421} = 0.649 \qquad & x_2 &= +\sqrt{0.00374} = 0.0612 \\ x_2 &= -\sqrt{0.421} = -0.649 \qquad & x_4 &= -\sqrt{0.00374} = -0.0612 \end{aligned}$$

[1] See, for instance, S. Timoshenko, and D. H. Young, "Engineering Mechanics—Statics," p. 185. McGraw-Hill Book Company, Inc., 1937.

Of these four roots the second and fourth have no physical significance, for they correspond to "negative" sags; and the first must be discarded, for it leads to a sag $f = 0.649a$, which is not small in comparison with a. The only root having physical significance is therefore the third, and the corresponding sag is

$$f = 6.06 \text{ ft}$$

3·4 Higher-degree Equations

a. Trial-and-error Solution

A sea mine, made out of steel in the shape of a hollow cubic box, is to have dimensions 2 ft on the side. The thickness of its walls must be determined so that the box will just float.

Calling a the side of the outer cube, W the specific weight of steel, and γ the specific weight of water and equating the weight of the mine to the weight of the displaced water, we find the following equation for the thickness x:

$$W[a^3 - (a - 2x)^3] = \gamma a^3$$

Since $\gamma/W = 1/7.85$ and $a = 2$, this equation becomes the *cubic equation* in x,

$$8x^3 - 24x^2 + 24x - 1.02 = 0 \qquad (a)$$

The formula for the solution of cubic equations, discovered by Tartaglia and published in more complete form by Cardano in 1545, is of practical use in theoretical derivations but is often abandoned in favor of other procedures, which are needed in any case for the solution of higher-degree equations. The formula for the solution of the quartic (fourth-degree) equation was found in the sixteenth century, but then for over 200 years the algebraists seemed unable to find formulas for the solution of the fifth or higher-degree equations, until in 1813 Ruffini and later Abel actually proved that it is *impossible* to solve rigorously equations of order higher than the fourth by means of a finite number of algebraic operations and roots.

The procedures devised to solve these equations are based on methods of successive approximations and require a rough value of the roots to start from. This value, as well as more accurate values of the roots, can very often be found by trial and error and graphical interpolation. Thus for $x = 0$ the left-hand member of Eq. (a) becomes equal to -1.02, while for $x = 1$ it equals 6.98. Due to the continuity of polynomial functions, one of the three roots of Eq. (a) lies between $x = 0$ and $x = 1$ and is much nearer to zero than to 1.

For $x = 0.1$ the left-hand member of Eq. (a) equals 1.148, and the graphical interpolation of Fig. 3·3 gives immediately a rough value of the root $x = 0.047$, for which the left-hand member of Eq. (a) equals $+0.056$. This process may be continued, if better accuracy is required,

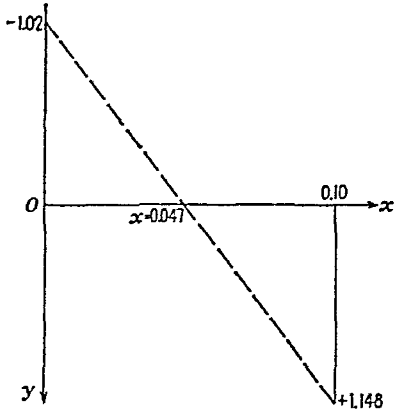


FIG. 3·3.

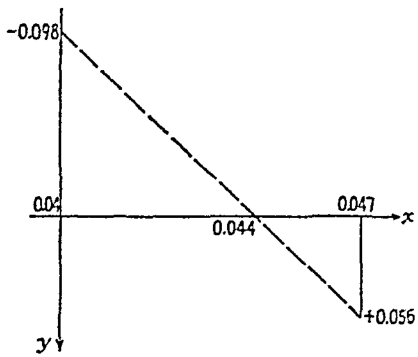


FIG. 3·4.

by trying, for example, $x = 0.04$, which gives a residual value of -0.098, and by using another interpolation (Fig. 3·4), which gives $x = 0.044$, a result correct to two significant figures.

b. Separation Intervals

A more systematic procedure for finding approximate values of all the real roots of an equation consists in studying the behavior of the function y, which equals its left-hand member. This will be demonstrated on the equation

$$y = x^3 - 2x^2 - x + 2 = 0 \tag{b}$$

by means of the properties of y and its derivatives up to the one linear in x, in this case, the second.

$$\begin{aligned} y' &= 3x^2 - 4x - 1 \\ y'' &= 6x - 4 \end{aligned}$$

The graph of y'' (Fig. 3·5a) is a straight line with a positive slope $m = 6$, crossing the x axis at $x = \frac{2}{3}$ and hence always negative to the left of $x = \frac{2}{3}$ and always positive to the right of $x = \frac{2}{3}$. Since y'' is the derivative of y', the rule of Sec. 2·7 a about the first derivative of increasing and decreasing functions indicates that y' decreases up to $x = \frac{2}{3}$ and increases from there on; and, since $y'(\frac{2}{3}) = -\frac{7}{3} < 0$, the graph of y'

must cross the x axis twice. The solution of the quadratic $y' = 0$ gives these intersections:

$$x = 1.55 \qquad x = -0.215$$

The graph of y' is therefore positive to the left of $x = -0.215$ and to the right of $x = 1.55$ and negative between $x = -0.215$ and $x = 1.55$ (Fig. 3·5b). Hence, applying again the rule of Sec. 2.7 a, the graph of y is increasing up to $x = -0.215$. decreasing from $x = -0.215$ to $x = 1.55$,

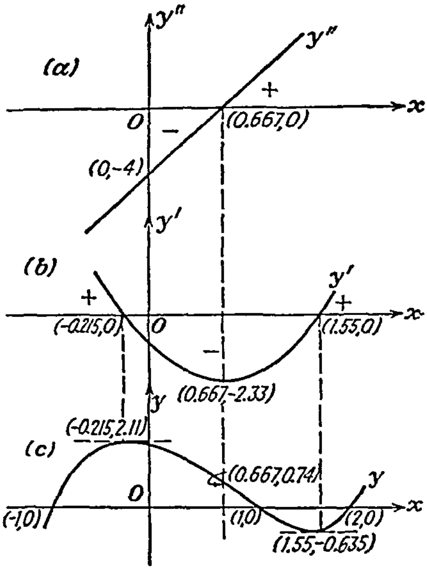


FIG. 3·5.

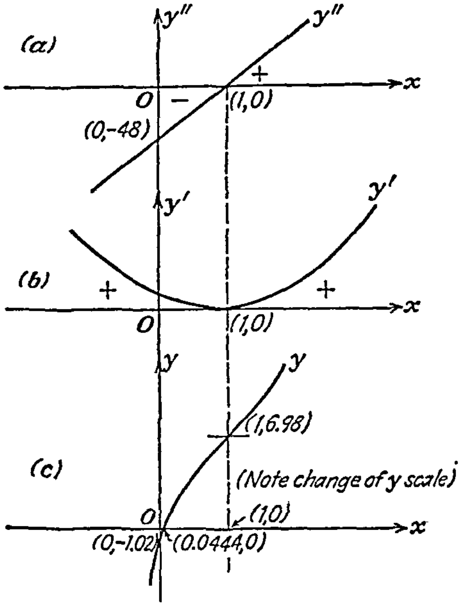


FIG. 3·6.

and increasing again from there on. Since the value of y is positive at $x = -0.215$ and negative at $x = 1.55$, the function y crosses the axis at three points (Fig. 3·5c), *i.e.*, Eq. (b) has three real roots, one to the left of -0.215, one between -0.215 and 1.55, and one to the right of 1.55.

An interval of the x axis, within which there falls one root of an equation and one root only, is called a *separation interval* for that root. A few trials show that, since y is negative at $x = -2$ and positive at $x = 3$, the following are separation intervals for the three roots of Eq. (b):

$$-2.00, -0.215; \qquad -0.215, 1.55; \qquad 1.55, 3.00$$

The three roots of Eq. (b) are actually

$$x_1 = -1 \qquad x_2 = 1 \qquad x_3 = 2$$

Figure 3·6 shows the graphs of y, y', and y'' for Eq. (a) of this section (page 97) and indicates that this equation has only one real root which

lies in the separation interval $x = 0$, $x = 1$. The point $x = 1$ is an inflection point for y since $y'' = 0$ at $x = 1$.

c. Computation of Real Roots by Successive Approximations: The Method of Chords

As soon as a separation interval x_0,x_1 has been found for a given root, we can close on the root by the procedure illustrated in Fig. 3·7, where y_i is the value of the left-hand member of the equation at $x = x_i$. The chord connecting the points $P_0(x_0,y_0)$ and $P_1(x_1,y_1)$ cuts the x axis at a point x_2 nearer to the root than x_1. Similarly, the new chord P_0P_2 cuts the x axis at x_3, which is nearer to the root than x_2, and the chord P_0P_3 gives a still better approximation x_4. This procedure can be repeated until we get as near to the root x as warranted by the accuracy of the problem, and the computations may be carried out analytically by means of the equation of the line through P_0P_1 [Eq. (2·3·3)],

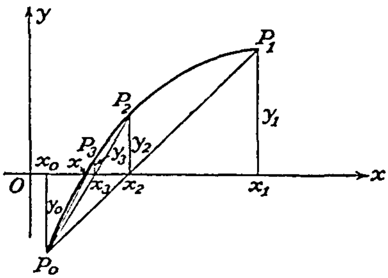


FIG. 3·7.

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0}$$

setting in this equation $y = 0$ to obtain the abscissa x_2 of its intersection with the x axis,

$$x_2 = x_0 - y_0 \frac{x_1 - x_0}{y_1 - y_0}$$

In general, if x_n is the nth approximation of the root and y_n the corresponding value of y, the $(n + 1)$th approximation is given by

$$x_{n+1} = x_0 - y_0 \frac{x_n - x_0}{y_n - y_0} \tag{3·4·1}$$

and the procedure is stopped as soon as, within the required degree of accuracy, two consecutive approximations of the root give the same value for x.

Table 3·1 shows the successive steps in the computation of the real root of Eq. (*a*) of this section (page 97), starting from the separation interval

$$x_0 = 0,\ y_0 = -1.02 \qquad x_1 = 1,\ y_1 = 6.98$$

It will be noticed that the successive approximations of x are all from above, *i.e.*, from the side of the first approximation x_1, as indicated by

TABLE 3·1

n	x_n	y_n	$x_n - x_0$	$y_n - y_0$	$-y_0 \dfrac{x_n - x_0}{y_n - y_0}$
1	1	6.98	1	8	0.1275
2	0.1275	1.667	0.1275	2.687	0.0484
3	0.0484	0.086	0.0484	1.106	0.0446
4	0.0446	0.004	0.0446	1.024	0.0444
5	0.0444	−0.001	0.0444	1.019	0.0444
6	0.0444				

the positive values of y_n. But, as in lines 4 and 5 of Table 3·1, it may happen that, because of unavoidable inaccuracies in the computations, two successive values of y_n have opposite signs. The separation interval thus reached cannot be shortened by the method of computation used, and the best value of x is found by a rough interpolation between the two last values obtained. In our case this gives $x = 0.04444$, which is one unit off in the last figure.

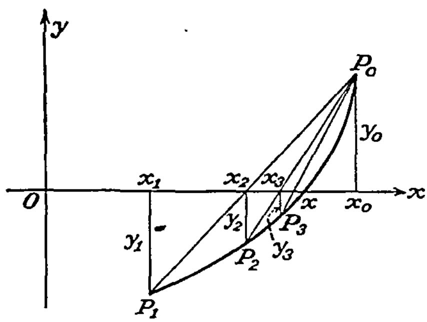


FIG. 3·8.

Comparison of the graph of Fig. 3·7 with the graph of Fig. 3·8 shows that the end of the separation interval to be used as initial point x_0 depends upon the curvature of y. When the curvature is positive (Fig. 3·8), x_0 is the right end of the interval and the successive approximations are from below; when the curvature is negative (Fig. 3·7) x_0 is the left end of the interval and the successive approximations are from above.

The method of chords may also be used to locate the real roots of nonalgebraic equations (see Sec. 3·5·b).

d. Synthetic Division

Knowing one of the real roots, say x_1, of an equation, the other roots can be more easily computed once the equation is divided through by the factor $(x - x_1)$. This is best done by *synthetic division*. The following scheme shows the operations involved in the process of dividing Eq. (b) of this section (page 98) by the factor $(x - r)$:

$$\begin{array}{c|cccc} & 1 & -2 & -1 & 2 \\ r & & 1r & r(r-2) & r(r^2-2r-1) \\ \hline & 1 & (r-2) & (r^2-2r-1) & \underline{\left| r^3 - 2r^2 - r + 2 \right.} \end{array}$$

The numbers of the first row are the coefficients of the equation.[1] Carry down to the third row the first coefficient 1, multiply by r, write the result in the second row, and add to the second coefficient -2. The product of $r - 2$ and r is written in the second row under the third coefficient -1 and added to it. The sum $r^2 - 2r - 1$ is again multiplied by r and written under the last coefficient 2, to which it is added to give $r^3 - 2r^2 - r + 2$, that is, the value of Eq. (b) at $x = r$.

For instance, at $x = -1$ we obtain

$$\begin{array}{r|rrr|r} & 1 & -2 & -1 & 2 \\ -1 & & -1 & 3 & -2 \\ \hline & 1 & -3 & 2 & \boxed{0} \end{array}$$

$x = -1$ is a root of Eq. (b), and the quadratic obtained by dividing Eq. (b) by the factor $(x + 1)$ has the coefficients of the third row of the scheme, *i.e.*, is equal to

$$x^2 - 3x + 2 = 0$$

as can be checked by long division.

To find the other two roots of Eq. (b), we compute the roots of this quadratic equation, which are found to be

$$x_2 = 1 \qquad x_3 = 2$$

Similarly, dividing Eq. (a) of this section (page 97) by its root $x_1 = 0.0444$, we get

$$\begin{array}{rrrrr} & 8 & -24 & +24 & -1.02 \\ 0.0444 & & 0.3552 & -1.0498 & 1.0190 \\ \hline & 8 & -23.6448 & 22.9502 & \boxed{-0.0010} \end{array}$$

$x = 0.0444$ is an *approximate* root of the equation, as shown by the remainder, which is small but not identically zero, and the equation can be *approximately* factored as

$$8x^3 - 24x^2 + 24x - 1.02 \doteq (x - 0.0444)(8x^2 - 23.64x + 22.95)$$

The other two roots of Eq. (a) are the roots of the quadratic equation

$$8x^3 - 23.64x + 22.95 = 0$$

i.e.,

$$x_2 = 1.48 + 0.828i \qquad x_3 = 1.48 - 0.828i$$

[1] The coefficients of the powers of x not appearing in the equation must be written as zeros. For instance, to divide $x^3 + 2x - 3$ by $(x - 1)$ we write

$$\begin{array}{rrrrr} & 1 & 0 & 2 & -3 \\ 1 & & 1 & 1 & 3 \\ \hline & 1 & 1 & 3 & \boxed{0} \end{array}$$

e. General Theorems

The methods outlined and demonstrated on third-degree equations in the preceding articles can be applied to the computation of the real roots of equations of any degree. The steps involved are as follows:

1. Locate separation intervals for each real root by trial and error or by curve behavior, solving in reverse order the equations of the successive derivatives of y.

2. Compute by successive approximations the values of the real roots $x_1, x_2, \ldots, x_n$ to the degree of accuracy required by the problem.

3. When the equation contains a single couple of conjugate complex roots, divide the equation by the binomials $(x - x_1)$, $(x - x_2)$, . . . , $(x - x_n)$ and thus derive the quadratic equation, whose two roots are the complex roots of the original equation. Solution of this equation gives the two complex roots. Thus an equation of the nth degree can be completely solved even if it has two complex roots.

When *all* the roots of a high-degree equation involving *more* than one couple of complex roots are desired, more powerful methods of solution must be resorted to. One such method is the *squaring the roots*, or *Graeffe's method*, whose explanation goes beyond the scope of this book.[1]

The following general theorems are often useful in connection with the determination of separation intervals and successive approximations of roots of equations with real coefficients:

1. Every algebraic equation of the nth degree has n roots, of which some may be repeated.

2. Every odd-degree equation has at least one real root, whose sign is opposite to the sign of its constant term.

3. Complex roots always appear in couples of complex conjugate numbers.

4. The number of positive roots of an equation is equal to the number of changes of sign of its coefficients or less than this by an even integer; the number of negative roots of an equation is given by the same rule after replacement of x by $-x$ in the equation (*Descartes's rule of signs*).[2]

For example, Eq. (*b*) may have either two or no positive roots, since

$$x^3 - 2x^2 - x + 2 = 0$$

presents two changes of sign. The corresponding equation with x changed into $-x$ is

$$-x^3 - 2x^2 + x + 2 = 0$$

[1] Excellent presentations of the method appear in R. E. Doherty, and E. G. Keller, "Mathematics of Modern Engineering," pp. 98*ff*., John Wiley & Sons, Inc., New York, 1936, and J. B. Scarborough, "Numerical Mathematical Analysis," pp. 198*ff*., Johns Hopkins Press, Baltimore, 1930.

[2] In this check, zero coefficients are skipped.

and has only one change of sign; hence Eq. (b) may have either one or no negative roots.

5. The following relations (*Newton's relations*) are satisfied in the nth-degree equation,

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with roots $x_1, x_2, \ldots, x_n$:

$$\begin{aligned} \sum_{i=1}^{n} x_i &= -\frac{a_{n-1}}{a_n} \\ \sum_{i,j=1}^{n} x_i x_j &= \frac{a_{n-2}}{a_n} \\ \sum_{i,j,k=1}^{n} x_i x_j x_k &= -\frac{a_{n-3}}{a_n} \\ &\cdots\cdots\cdots \\ x_1 x_2 x_3 \cdots x_n &= (-1)^n \frac{a_0}{a_n} \end{aligned} \tag{3·4·2}$$

In the preceding formulas $\sum_{i=1}^{n} x_i$ is the sum of all the roots, $\sum_{i,j=1}^{n} x_i x_j$ is the sum of the products of the roots taken two by two, $\sum_{i,j,k=1}^{n} x_i x_j x_k$ is the sum of the products of the roots taken three by three, etc. For instance, in the cubic equation $a_3x^3 + a_2x^2 + a_1x + a_0 = 0$ with roots x_1, x_2, and x_3, we have

$$\begin{aligned} x_1 + x_2 + x_3 &= -\frac{a_2}{a_3} \\ x_1x_2 + x_1x_3 + x_2x_3 &= \frac{a_1}{a_3} \\ x_1x_2x_3 &= -\frac{a_0}{a_3} \end{aligned}$$

3·5 Transcendental Equations and Newton's Method

a. *Transcendental Equations*

The mechanization of the means of production has consistently reduced the number of workers needed to mine a given quantity of coal, while the amount of coal needed by industry has consistently increased during the last 60 years. In Fig. 3·9, curve y_1 gives the number of man-

hours needed to dig 1 ton of coal versus the time t, in years, between 1890 and 1940, while curve y_2 gives the total amount of coal mined each year for the same period. Mechanization and industrial needs are therefore contrasting factors in the employment of coal miners, and it is desired to find in what year the coal industry employed or will employ the largest number of miners, if the trend of the last 60 years persists.

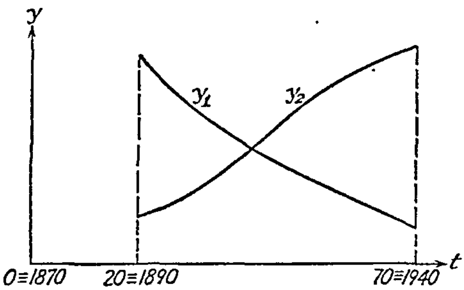


FIG. 3·9.

The two curves of Fig. 3·9 can be well approximated by the functions

$$y_1 = \frac{k_1}{t^c} \qquad y_2 = \frac{k_2}{1 + e^{a-bt}}$$

where k_1, k_2, a, b, and c are all positive constants ($a > c > 1$) and the origin of time t is taken at the year 1870. By means of y_1 and y_2 the number of man-hours employed in any given year becomes

$$y = y_1 \times y_2 = \frac{k}{t^c(1 + e^{a-bt})}$$

where $k = k_1k_2$. Since y is a continuous function, y' must be zero when y is maximum,

$$y' = -k\,\frac{(1 + e^{a-bt})ct^{c-1} - t^cbe^{a-bt}}{[t^c(1 + e^{a-bt})]^2} = 0$$

and, since the denominator of y' cannot be infinite, its numerator must be equal to zero,

$$t^{c-1}[c(1 + e^{a-bt}) - bte^{a-bt}] = 0$$

This equation has an obvious root, $t = 0$, which can be immediately discarded because $t = 0$ falls outside the actual range of times considered in the problem, which starts at $t = 20$ (1890). The other roots are then given by the expression in the bracket, which can be written as

$$e^{a-c}e^{c-bt}(c - bt) + c = 0$$

or, letting

$$c - bt = -z \qquad A = \frac{e^{a-c}}{c} \tag{a}$$

as

$$Az = e^z \tag{b}$$

Equation (b) is a nonalgebraic, or *transcendental*, equation in the unknown z. Formulas for the solution of transcendental equations are not avail-

able, and the best method for the evaluation of their roots is the location of separation intervals and the use of successive approximations. Separation intervals are best located graphically in this case.

If we call w_1 and w_2 the left- and right-hand members, respectively, of Eq. (b), the solution of Eq. (b) is reduced to the location of the intersections of the straight line

$$w_1 = Az$$

and the exponential curve

$$w_2 = e^z$$

These intersections are real (Fig. 3·10) if and only if the slope A of the straight line is larger than the slope of the tangent to w_2 from the

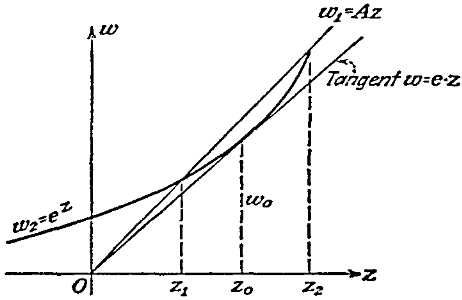


FIG. 3·10.

origin. To check this, calling z_0, w_0 the coordinates of the point of tangency and remembering that the slope of w_2 at z_0 is

$$\left.\frac{dw_2}{dz}\right]_{z=z_0} = e^{z_0}$$

we write the equation of the tangent to w_2 through the origin, which by Eq. (2·7·1) is

$$w = e^{z_0}z$$

Since at $z = z_0$ the curve $w_2 = e^z$ and the tangent must have the same ordinate,

$$e^{z_0}z_0 = e^{z_0}$$

the abscissa z_0 of the point of tangency equals 1 and the slope of the tangent equals e. The condition for Eq. (a) to have real roots is therefore

$$A > e$$

In the present problem we have, from statistical data,

$$a = 3.57 \qquad b = 0.093 \qquad c = 1.06$$

hence

$$A = \frac{e^{3.57-1.06}}{1.06} = 11.61 > e$$

and Eq. (*b*) has two real roots. From the graph of Fig. 3·10, $z_1 = 0.10$ and $z_2 = 3.8$ are rough values of these roots, by means of which the first of Eqs. (*a*) gives $t_1 = 12$ years, $t_2 = 52$ years. The graph of the function y shows that y is minimum at $t_1 = 13$ and maximum at $t_2 = 52$; hence the employment of coal miners was at a maximum in

$$1870 + 52 = 1922$$

The accuracy of our problem requires not more than two figures in the roots, since statistical tables give employment figures year by year. But a better approximation can be achieved, whenever needed, by successive approximations, as explained for the case of algebraic equations in Sec. 3·4 *c*.

Table 3·2 shows the computation of the root z_1 to three significant figures by the method of chords applied to the equation

$$w = e^z - 11.61z = 0$$

with separation interval $z_1 = 0.10, w_1 = -0.0558; z_0 = 0.09, w_0 = 0.0493$.

TABLE 3·2

n	z_n	w_n	$z_n - z_0$	$w_n - w_0$	$-w_0 \dfrac{z_n - z_0}{w_n - w_0}$
1	0.10	−0.0558	0.0100	−0.1051	+0.0053
2	0.0947	−0.0002	0.0047	−0.0495	+0.000047
3	0.0947	−0.0002			

b. Newton's Method

Another method of successive approximations, *which can also be used in connection with both algebraic and transcendental equations*, is *Newton's method*, or the *method of tangents*, illustrated in Fig. 3·11. If x_0 is a rough approximation of a root and y_0 is the corresponding value of the left-hand member y of an equation, the intersection of the tangent to the curve y from P_0 with the x axis is a better approximation, x_1, of the root. Drawing the tangent to the curve y at $P_1(x_1,y_1)$, we find the new intersection x_2, which is nearer to the root than x_1; and the process can be similarly repeated until two successive approximations give, within the accuracy required by the problem, identical results.

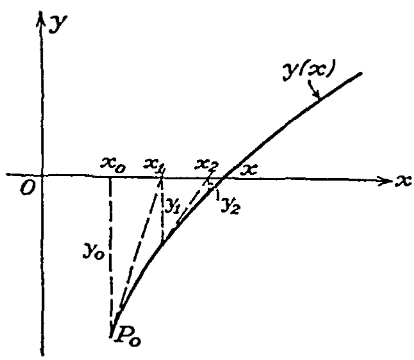


FIG. 3·11.

The procedure can be carried out analytically by writing the equation of the tangent to a curve at a point $x_0.y_0$ [Eq. (2·7·1)],

$$y - y_0 = y'_0(x - x_0)$$

and by setting $x = 0$, to get the intersection x_1,

$$x_1 = x_0 - \frac{y_0}{y'_0}$$

In general, if x_n is the nth approximation of a root and y_n the corresponding value of the left-hand member of the equation, the $(n + 1)$th approximation is given by

$$x_{n-1} = x_n - \frac{y_n}{y'_n} \tag{3·5·1}$$

To apply Newton's method to the smaller root of

$$w = e^z - 11.61z = 0$$

we compute w',

$$w' = e^z - 11.61$$

and start the computations at $z_1 = 0.10$, $w_1 = -0.0558$. Table 3·3 shows the results of the computation of the root to three significant figures. In most cases Newton's method is more rapidly convergent than the method of chords.

TABLE 3·3

n	z_n	w_n	w'_n	$\frac{-w_n}{w'_n}$
1	0.10	−0.0558	−10.5048	−0.0053
2	0.0947	−0.0002	−10.5107	−0.00002
3	0.0947	−0.0002		

Upon applying the same method to Eq. (a) of Sec. 3·4 a (page 97)

$$y = 8x^3 - 24x^2 + 24x - 1.02$$
$$y' = 24x^2 - 48x + 24$$

the solution is obtained as shown in Table 3·4, with $x_0 = 0$.

TABLE 3·4

n	x_n	y_n	y'_n	$\frac{-y_n}{y'_n}$
0	0	−1.02	24	0.0425
1	0.0425	−0.04274	22.00344	0.001942
2	0.04444	−0.000014	21.91428	0.000006
3	0.044446	−0.000004		

Newton's method should be used with caution if the first approximation is near a stationary point ($y' = 0$) or an inflection point ($y'' = 0$), since the second approximation may be worse than the first in these cases (see Fig. 3·12).

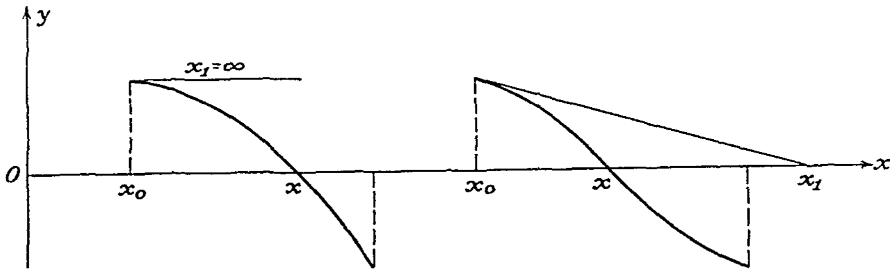


FIG. 3·12.

Either end of the separation interval can be used as a starting point for Newton's method; but, as shown in Fig. 3·13, *it is always advantageous to use as x_0 the end at which the sign of y is the same as the sign of y''* (sign of the curvature).

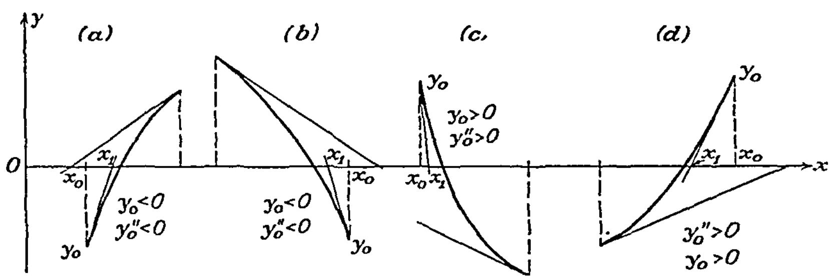


FIG. 3·13.

When the computation of y'_n is cumbersome, a simplified Newton's formula may be used, in which the first value of the slope y'_0 is used in *all* approximations,

$$x_{n+1} = x_n - \frac{y_n}{y'_0} \tag{3·5·2}$$

This procedure is slower in convergence, but the time spent in computing one or two additional approximations may be shorter than the time required for the evaluation of y'_n. Table 3·5 shows the application of Eq. (3·5·2) to Eq. (*a*) of Sec. 3·4 *a*, with $x_0 = 0$, $y'_0 = 24.00$.

Newton's method may also be used in conjunction with the method of chords to obtain lower and upper bounds of the roots, since the two procedures give always opposite bounds, as shown in Fig. 3·14.

TABLE 3·5

n	x_n	y_n	$\frac{-y_n}{y'_0}$
0	0	−1.02	0.0425
1	0.0425	−0.04274	0.001780
2	0.04428	−0.003643	0.000151
3	0.044431	−0.000333	0.000013
4	0.044444	−0.000048	0.000002
5	0.044446	−0.000004	

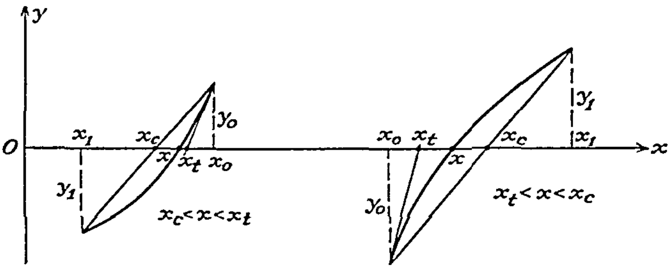


FIG. 3·14.

Problems

1. Solve the following equations for the unknown x:

(a) $\dfrac{4x-2}{x-1} = 1$ (b) $4x - 3 = 7a$

(c) $\dfrac{x-a}{a} - \dfrac{ax-b}{b} = \dfrac{4a-bx}{5}$ (d) $\dfrac{5x}{4} - \dfrac{3x}{2} = \dfrac{1}{6} - \dfrac{2x}{3}$

(e) $\dfrac{3x}{b(x-a)} = 7$ (f) $\dfrac{x^2-4}{x-2} = \dfrac{x^2-x-6}{x-3}$

Solve for x the following equation:

$$a^x + ba^{x-1} = 1$$

At what speed will an atomic particle double its "rest" mass? *Hint:*

$$m = \frac{m_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

where m_0 = rest mass
m = mass at velocity v,
c = speed of light

4. A pipe line 6 in. in diameter delivers 300 barrels of oil per hour. At what linear speed is the oil flowing in the pipe? (1 barrel = 42 gal.)

5. A d-c voltage of 100 volts is impressed on a resistive network consisting of 2 resistors of 10 ohms and 40 ohms in parallel. What current is flowing in each resistance? What is the equivalent resistance of the circuit?

6. A certain amount of gas is compressed isothermally (PV = const) until its original volume of 2 cu ft is reduced to 0.7 cu ft. By what percentage has the pressure increased?

7. A circular steel bar 2 ft long and 3 in. in diameter is elongated 0.01 in. by a uniform tension. What is the force in the bar if Young's modulus $E = 30 \times 10^6$ psi? *Hint:* Unit elongation = stress/E.

8. If a man can do a job in 4.5 days, what percentage of the job can he do in x days?

9. A right circular cone with a base 2 ft in diameter has a volume of 1000 cu in. What is the vertex angle of the cone?

10. A boat can travel 8 knots in still water. It steamed 24 nautical miles against a strong wind and 40 miles with the wind abaft in the same length of time. What was the speed of the wind?

11. Find the roots of the following quadratic equations:

(a) $x^2 - 5x + 6 = 0$ (b) $x^2 + 2x + 3 = 0$
(c) $x^2 - ax + b + 3 = 0$ (d) $4.21x^2 + 3.07x - 2.14 = 0$
(e) $x^2 + 2bx + c = 0$ (f) $2x^2 - 12x + 18 = 0$

12. A quadratic equation has the sum of its roots equal to 7 and the product equal to 4. Write the equation, and compute the roots.

13. A quadratic equation has the difference of its roots equal to $6i$ and the product equal to 25. Write the equation, and compute the roots.

14. The difference between the inner and the outer volume of a hollow cubical box is 199 cu in. The outer edge is 1 in. longer than the inner one. Find the inner and outer dimensions of the cube.

15. A body of mass 2 slugs falls under the action of gravity. How long will it take to fall 100 ft if its initial speed is 20 ft per sec down? (Neglect air resistance.)

16. A cloth manufacturer finds that a rectangular piece shrinks 5 per cent in width and 8 per cent in length when processed. If its total loss in perimeter is 3 ft and its loss in area 10 sq ft, what were the original dimensions of the cloth?

17. Train A runs 10 mph faster than train B and takes 3 hr less to cover 200 miles. Find the speed of each train.

18. A new lathe can do a certain job in $1\frac{1}{2}$ hr less than an old one. By using both lathes, the job can be completed in $\frac{3}{4}$ hr. How long would it take each lathe to do the job?

19. A retail dealer bought a number of radios for \$2500. Owing to storage and mishandling, 5 radios were ruined beyond repair. The dealer sold the remainder at a profit of \$20 each, thus making \$2400 on the whole transaction. How many radios did he buy, and how much did he originally pay for each radio?

20. By cutting the price per crate of oranges 20 cents, a grocer finds that he can sell 5 crates more than formerly. In both cases the total sale price is \$110. What was the former price of the oranges per crate?

21. Solve the following equation for x, and check by substitution:

$$\sqrt{4x + 8} - \sqrt{x + 2} = \sqrt{3x - 2}$$

22. Find all the roots of the following equations:

(a) $4x^4 - 1 = 0$ (b) $x^4 + 16 = 0$
(c) $x^4 - 5x^2 + 6 = 0$ (d) $x^4 - 2x^2 - 3 = 0$
(e) $3.12x^4 - 2.24x^2 + 7.12 = 0$ (f) $0.24x^4 - 3.62x^2 - 6.37 = 0$

23. A cable of length 200 ft is suspended between two supports on a horizontal plane. What is the sag if the supports are 190 ft apart?

24. Sketch the graph of the following curves, and determine separation intervals for their real roots:

(a) $x^3 - 4x + 10 = 0$	(b) $x^3 - 4x^2 + 7x - 3 = 0$
(c) $x^3 + x - 3 = 0$	(d) $x^3 + 2x^2 - x + 7 = 0$
(e) $x^3 + 2x - 1 = 0$	(f) $x^3 - 15x - 5 = 0$
(g) $4.1x^3 - 3.2x^2 - 7.3x + 11.4 = 0$	(h) $2.5x^4 - 7.1x - 1 = 0$
(i) $x^3 + x + 1 + \frac{1}{x} = 0$	(j) $x^4 - 3x + 2 = 0$

25. Evaluate to 2 significant figures, by trial and error, the smallest real root of Eqs. (a), (c), (e), (g), and (j) in Prob. 24.

26. Compute to 3 significant figures, by the method of chords, the value of the real roots of Eqs. (a) to (e) in Prob. 24, and evaluate the complex roots by solution of the corresponding quadratic equation.

27. Compute to 3 significant figures, by Newton's method, the value of the real roots of Eqs. (f) and (g) in Prob. 24, and evaluate the complex roots by solution of the corresponding quadratic equation.

28. What are the sum and product of the roots in each equation of Prob. 24?

29. How many positive and negative roots may the equations of Prob. 24 have?

30. Write the cubic equation whose roots are in the ratio 1:2:3.

31. Show that $x^3 + ax + c = 0$ can have but 1 real root if $a > 0$. *Hint:* Assume that the equation has 3 real roots, $x_1 > x_2 > x_3$, and reach a contradiction.

32. The van der Waals equation

$$v^3 - \left(b + \frac{RT}{P}\right) v^2 + \frac{a}{P} v - \frac{ab}{P} = 0$$

giving the relation between the volume v, the pressure P, and the absolute temperature T for a nonideal gas, has 3 real roots v when the temperature is lower than the critical temperature T_c. At the critical temperature T_c and at the critical pressure P_c the equation has 3 coincident roots v_c. Express the constants a, b, and R in terms of v_c, P_c, and T_c.

33. A pressure vessel, made up of a cylindrical shell closed by a flat top and a hemispherical bottom, has a volume of 200 cu ft. If the height of the cylinder is 30 ft, what is the radius of the hemisphere to 3 significant figures? (Compute by the method of chords.)

34. A thin spherical shell of external radius 12 ft has a volume $^{15}\!/_{16}$ that of a solid sphere of the same radius. Determine by Newton's method the thickness of the shell to 3 significant figures.

35. The relationship between the pressure P and the volume V of a nonideal gas at an absolute temperature T is given by

$$P = \frac{RT}{V - b} - \frac{a}{V^2}$$

where R, a, and b are constants. At a temperature of 27°C a certain container of CO_2 is at a pressure of 4 atm. Compute to 2 significant figures the volume of the container in cubic centimeters per mole, assuming

$$a = 3.60 \times 10^6 \left(\frac{\text{cc}}{\text{mole}}\right)^2 \text{atm}$$
$$b = 42.8 \text{ cc per mole}$$
$$R = 82 \frac{\text{cc-atm}}{\text{mole-°K}}$$

36. The cosine of a certain angle α less than $\pi/2$ is 3 times the cosine of an angle 3α. What is value of the angle α to the nearest minute?

37. Solve the following equation for x to 3 significant figures by Newton's method:

$$0.589 - x + \ln(2x) = 0$$

38. In fluid mechanics the friction factor f is related to the Reynolds number R by

$$\frac{1}{\sqrt{f}} = 2 \log_{10} (R \sqrt{f}) - 0.8$$

Determine the value of f for $R = 5.25 \times 10^6$ to 3 significant figures.

39. Determine to 2 significant figures the value of the hydraulic radius R for which the Chezy factor C will be the same whether the Manning or the Nikuradze formulas are used.

Manning: $C = 106.4R^{1/6}$
Nikuradze: $C = 112.4 + 32.1 \log_{10} R$

40. Calculate the smallest positive roots of the following equations to 2 significant figures:

(a) $\cos x - x = 0.726$ (b) $\sin x - x^2 = 0.213$
(c) $e^{-x} - \sin x = 0$ (d) $x + \tan x = 3.61$

41. From the mid-point of one of the smaller sides of a rectangle 2 by 4 ft an arc of a circle is drawn such that it divides the rectangle into 2 equal areas. Find the radius of the circle to 2 significant figures by trial and error.

42. A segment of a circle of radius 4 ft has an area of 2 sq ft. What is the apothem? (Solve by trial and error to 2 significant figures.)

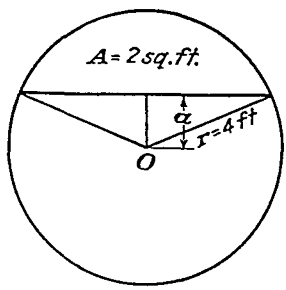


FIG. 3·15.

43. The voltage in a condenser at a time t is given by

$$V = E(1 - e^{-t/RC})$$

Compute in terms of RC the time it will take the voltage to reach 84 per cent of its final value.

44. The logarithmic-mean temperature difference T_m is defined by

$$T_m = \frac{T_2 - T_1}{\ln (T_2/T_1)}$$

If $T_m = 2810$°K and $T_2 = 6050$°K, find T_1.

45. If the amplification factor of n cascaded stages of double-tuned transformers is equal to $K\sqrt{2^{1/n} - 1}$, find the number of stages for maximum gain, remembering that the over-all amplification of n stages is the nth power of the amplification of one stage. Assume $K = 4$.

CHAPTER IV

THE NUMERICAL SOLUTION OF SIMULTANEOUS LINEAR ALGEBRAIC EQUATIONS

4·1 Introductory Example

Table 4·1 gives the number of hours of drill, punch, and lathe needed to manufacture 1000 items each of three kinds of products A, B, and C (including short delays and switch-over times). The manager of the

TABLE 4·1

A	B	C	
2	2	2	Drill
1	3	2	Punch
4	2	1	Lathe

manufacturing plant wants to know how many items of each kind can be manufactured in an 8-hr day by running the machines without interruptions.

If we call x, y, and z the number of items A, B, and C, in thousands, manufactured in a day, in manufacturing them the drill will work, all together, $2x + 2y + 2z$ hr; and since the drill is to run 8 hr,

$$2x + 2y + 2z = 8$$

Similarly, the punch works $x + 3y + 2z$ hr and the lathe $4x + 2y + z$ hr in an 8-hr day. Hence the three unknown quantities x, y, and z are found to be roots of the *system of linear equations*

$$\left.\begin{aligned} 2x + 2y + 2z &= 8 \\ x + 3y + 2z &= 8 \\ 4x + 2y + z &= 8 \end{aligned}\right\} \qquad (a)$$

Equations (a) furnish one of the numerous examples of linear systems that are encountered in practically all fields of applied mathematics. Their solution is of the utmost importance to the engineer, and it has received the attention of some of the greatest mathematicians.

System (a) may be easily solved by "elimination" as follows: Subtract the second from the first equation, obtaining

$$x - y = 0 \qquad \text{or} \qquad y = x \qquad (b)$$

Substitute this value for y in the second and third equations.

$$\left.\begin{aligned} 4x + 2z &= 8 \\ 6x + z &= 8 \end{aligned}\right\} \qquad (c)$$

Multiply the second equation by 2, and subtract from it the first.

$$\begin{array}{r} 12x + 2z = 16 \\ 4x + 2z = 8 \\ \hline 8x = 8 \end{array}$$

Therefore $x = 1$; from Eq. (b), $y = 1$; from the second of Eqs. (c), $z = 2$. While one or more negative roots would have indicated the physical impossibility of a solution, the present problem is solvable since all the roots of Eqs. (a) are positive.

More systematic procedures for the solution of simultaneous equations will be demonstrated in the following sections.

4·2 Determinants

a. Second-order Determinants

The elimination procedure used in the preceding section can be systematized as follows: Consider the system of two equations in the unknowns x and y,

$$\left.\begin{aligned} \text{(I)} \quad a_1x + b_1y &= c_1 \\ \text{(II)} \quad a_2x + b_2y &= c_2 \end{aligned}\right\} \qquad (4\cdot2\cdot1)$$

in which the constants appear in the right-hand member of the equations. If we multiply Eq. (I) by b_2, Eq. (II) by b_1, and subtract Eq. (II) from Eq. (I), we obtain the new equation

$$(a_1b_2 - a_2b_1)x = c_1b_2 - c_2b_1$$

containing the single unknown x. The unknown y has thus been eliminated between Eqs. (I) and (II), and the value of x may be immediately computed.

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}$$

Similarly, multiplying Eq. (I) by a_2, Eq. (II) by a_1, and subtracting Eq. (II) from Eq. (I), we eliminate x and the value of y becomes

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

The values of the unknowns x and y are thus found to be given by two fractions with identical denominators, whose numerators and denominators are differences of products of the type $A \times D - B \times C$. Such

difference of products is called a *determinant of the second order* and is conveniently represented by the symbol

$$\begin{vmatrix} A & C \\ B & D \end{vmatrix} = AD - BC \tag{4·2·2}$$

The symbol of a determinant is equal to, but should not be confused with, the symbol of the absolute value of a number.

By Eq. (4·2·2) the roots of Eqs. (4·2·1) can be written in terms of second-order determinants as

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \tag{4·2·3}$$

The solutions of Eqs. (4·2·1) can now be written directly by inspection, if we notice that

1. The denominators of Eqs. (4·2·3) are the determinant D in which the coefficients of the unknowns, a_1,a_2 and b_1,b_2, appear in the same locations as in the system (4·2·1). This determinant is called the determinant of the coefficients of the system or the *determinant of the system.*

2. The numerators of x and y in Eqs. (4·2·3) are obtained from the determinant of the coefficients D by substitution of the constants c_1 and c_2 for the coefficients a_1,a_2 and b_1,b_2 of the unknowns x and y, respectively.

For example, the roots of the system

$$\begin{aligned} 2x + y &= 4 \\ x - 2y &= -3 \end{aligned}$$

are written directly in determinantal form as

$$x = \frac{\begin{vmatrix} 4 & 1 \\ -3 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{4(-2) - (-3)1}{2(-2) - 1(1)} = \frac{-5}{-5} = 1$$

$$y = \frac{\begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{2(-3) - 1(4)}{2(-2) - 1(1)} = \frac{-10}{-5} = 2$$

b. Higher-order Determinants

A system of three equations (I), (II), and (III) in the three unknowns x, y, and z can be similarly solved by eliminating one unknown, say x, first between Eqs. (I) and (II) and again between Eqs. (I) and (III), and thus obtaining a system of two equations in y and z only, which can

be solved by elimination, as was done in the preceding section (page 115). Once y and z are known, any of the three original equations will give x. If this procedure, applied literally, is extended to a system of n equations in n unknowns, it is found that the roots of the system are given by ratios, which can be conveniently written in determinantal form by means of the following rule:

Rule: The system of n equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1i}x_i + \cdots + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2i}x_i + \cdots + a_{2n}x_n = c_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{ni}x_i + \cdots + a_{nn}x_n = c_n \end{array}\right\} \tag{4·2·4}$$

has n roots $x_1, x_2, \ldots, x_n$ given by

$$x_i = \frac{\begin{vmatrix} a_{11} & a_{12} & \cdots & c_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & c_2 & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & c_n & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{ni} & \cdots & a_{nn} \end{vmatrix}} \qquad (i = 1, 2, \ldots, n) \tag{4·2·5}$$

The symbols in the numerator and denominator of Eqs. (4·2·5) are called *determinants of order n* and are written by inspection according to rules 1 and 2 of Sec. 4·2 *a*. The denominator is the determinant of the coefficients D. The numerator is obtained by substituting in D the constants c for the coefficients of the unknown x_i. (Notice that the *constants c have the sign with which they appear in the right-hand* member of the equations.) The numbers a_{ij} $(i,j = 1, 2, \ldots, n)$ are called the *elements* of the determinant. The elements with the same first subscript i form the *ith row* of the determinant. The elements with the same second subscript j form the *jth column* of the determinant. Hence the element a_{ij} is located at the intersection of the ith row with the jth column.

From the literal solution of the system (4·2·4) it is found that the determinant of the coefficients D is equal to

$$D = \Sigma(a_{1i_1}a_{2i_2} \cdots a_{ni_n})(-1)^r \tag{4·2·6}$$

where the sum is extended to all the $n!$ different products[1] of n elements

[1] The symbol $n!$ (n factorial) stands for the product $1 \times 2 \times 3 \times \cdots \times n$, and by definition $0! = 1$.

obtained by taking one element from each row and from each column of the determinant.[1] The exponent m is the number of interchanges of adjacent elements needed to bring the sequence of second subscripts $i_1, i_2, \ldots, i_n$ to the normal order $1, 2, \ldots, n$. For instance, in a second-order determinant

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

one of the terms of the sum is $a_{11}a_{22}$, in which the second subscripts are already in the normal order 1, 2, and therefore $m = 0$; the only other term is $a_{12}a_{21}$; and since the second subscripts 2, 1 must be interchanged once to bring them to the normal order, $m = 1$. Hence

$$D = (-1)^0 a_{11}a_{22} + (-1)^1 a_{21}a_{12} = a_{11}a_{22} - a_{21}a_{12}$$

which checks with the definition (4·2·2).

c. Laplacian Expansion

Since Eq. (4·2·6) is at best a cumbersome method for the evaluation of a determinant of order higher than 2, other equivalent procedures, or "expansions," have been devised, better suited to numerical or literal computations.

If we strike from a determinant of order n the row and column meeting at the element a_{ij}, we obtain a determinant of order $n - 1$ called the *minor* M_{ij}. The *signed minor* $(-1)^{i+j}M_{ij}$, which is $+M_{ij}$ when $i + j$ is even and $-M_{ij}$ when $i + j$ is odd, is called the *cofactor* of a_{ij}.

It may be helpful to notice that the sign of the cofactor $(-1)^{i+j}M_{ij}$ is given by the following alternating scheme:

$$\begin{vmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

The value of a determinant can be expressed in terms of cofactors by the following sum:

$$\sum_{j=1}^{n} a_{ij}(-1)^{i+j}M_{ij} \quad (i = 1, 2, \ldots, n) \qquad (4·2·7)$$

Equation (4·2·7) is called the *Laplacian expansion* of a determinant by the elements of the ith row and can be stated in words as follows: *The*

[1] See, for instance, M. Bôcher, "Introduction to Higher Algebra," p. 24, The Macmillan Company, New York, 1935.

value of a determinant is equal to the sum of the products of the elements of any one given row by their respective cofactors.

For instance, a determinant of the third order is given by any of the following sums:

$$
\begin{aligned}
D_3 &= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}\\
D_3 &= -a_{21}M_{21} + a_{22}M_{22} - a_{23}M_{23}\\
D_3 &= a_{31}M_{31} - a_{32}M_{32} + a_{33}M_{33}
\end{aligned}
$$

Since the M_{ij} appearing in the Laplacian expansion are determinants of order $n - 1$, they can be evaluated by means of Eq. (4·2·7) in terms of determinants of order $n - 2$; thus by successive steps the calculation is reduced to second-order determinants, computable by Eq. (4·2·2), which is a particular case of Eq. (4·2·7).

Example of fourth-order determinant expanded by the elements of the first row:

$$
D = \begin{vmatrix} 2 & 1 & 1 & 1\\ -1 & 2 & 1 & 2\\ 1 & 1 & 1 & 2\\ 2 & 2 & 1 & 1 \end{vmatrix} = +2\begin{vmatrix} 2 & 1 & 2\\ 1 & 1 & 2\\ 2 & 1 & 1 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 & 2\\ 1 & 1 & 2\\ 2 & 1 & 1 \end{vmatrix}
$$

$$
+1\begin{vmatrix} -1 & 2 & 2\\ 1 & 1 & 2\\ 2 & 2 & 1 \end{vmatrix} - 1\begin{vmatrix} -1 & 2 & 1\\ 1 & 1 & 1\\ 2 & 2 & 1 \end{vmatrix}
$$

$$
= 2\left(2\begin{vmatrix} 1 & 2\\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2\\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 1\\ 2 & 1 \end{vmatrix}\right) - 1\left(-1\begin{vmatrix} 1 & 2\\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2\\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 1\\ 2 & 1 \end{vmatrix}\right)
$$

$$
+1\left(-1\begin{vmatrix} 1 & 2\\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2\\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 1\\ 2 & 2 \end{vmatrix}\right) - 1\left(-1\begin{vmatrix} 1 & 1\\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 1\\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1\\ 2 & 2 \end{vmatrix}\right)
$$

$$
= 2[2(-1) - 1(-3) + 2(-1)] - 1[-1(-1) - 1(-3) + 2(-1)]
$$

$$
+ 1[-1(-3) - 2(-3) + 2(0)] - 1[-1(-1) - 2(-1) + 1(0)]
$$

$$
= 2(-1) - 1(2) + 1(9) - 1(3) = 2
$$

By means of Eqs. (4·2·5) and the Laplacian expansion (4·2·7) the roots of the system (a) of Sec. 4·1 can now be computed directly.

$$
x = \frac{\begin{vmatrix} 8 & 2 & 2\\ 8 & 3 & 2\\ 8 & 2 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 2 & 2\\ 1 & 3 & 2\\ 4 & 2 & 1 \end{vmatrix}} = \frac{8(-1) - 2(-8) + 2(-8)}{2(-1) - 2(-7) + 2(-10)} = \frac{-8}{-8} = 1
$$

$$y = \frac{\begin{vmatrix} 2 & 8 & 2 \\ 1 & 8 & 2 \\ 4 & 8 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 2 & 2 \\ 1 & 3 & 2 \\ 4 & 2 & 1 \end{vmatrix}} = \frac{2(-8) - 8(-7) + 2(-24)}{-8} = \frac{-8}{-8} = 1$$

$$z = \frac{\begin{vmatrix} 2 & 2 & 8 \\ 1 & 3 & 8 \\ 4 & 2 & 8 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 2 \\ 1 & 3 & 2 \\ 4 & 2 & 1 \end{vmatrix}} = \frac{2(8) - 2(-24) + 8(-10)}{-8} = \frac{-16}{-8} = 2$$

d. Properties of Determinants

The previous examples show that the computation of a determinant becomes very cumbersome, even by the Laplacian expansion, as soon as its order is higher than 3. The following properties, which will be demonstrated for determinants of the second order *but are valid for any order*, may simplify the expansion of a determinant:

1. The value of a determinant is not changed if its rows are written as columns and its columns as rows.

$$\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = -2 \qquad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$$

The interchange of rows with columns allows the Laplacian expansion of a determinant *by the elements of a given column*

$$D = \sum_{i=1}^{n} a_{ij}(-1)^{i+j}M_{ij} \quad (j = 1, 2, \ldots, n) \qquad (4\text{-}2\text{-}7a)$$

2. A factor common to all the elements of a given row (or column) can be divided out and placed as a factor of the new determinant.

$$\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} = -10 = 2\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 2(-5) = -10$$

Conversely, to multiply a determinant by a factor, each element of any *one* row or column must be multiplied by the factor.

3. A determinant is changed in sign by interchanging any two rows or columns (not necessarily adjacent).

$$\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} = -\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} = -\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = -10$$

4. A determinant is equal to zero if any two of its rows or columns have proportional elements.

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0$$

5. A determinant is not changed in value, if we add to the elements of any row (or column) the elements of another row (or column) multiplied by a factor. For example, given

$$\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = -10$$

adding to the second row the elements of the first row multiplied by (-2), we get

$$\begin{vmatrix} 2 & 3 \\ 0 & -5 \end{vmatrix} = -10$$

This property is often used to reduce to zero as many elements of the determinant as conveniently possible. For instance, in the determinant in the numerator of the root x of Sec. 4·2 c we may factor an 8 from the first column and subtract the second row from the first, after which the expansion is reduced to a single term

$$\begin{vmatrix} 8 & 2 & 2 \\ 8 & 3 & 2 \\ 8 & 2 & 1 \end{vmatrix} = 8\begin{vmatrix} 1 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 8\begin{vmatrix} 0 & -1 & 0 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 8(1)\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 8(1)(-1) = -8$$

Unfortunately the determinants encountered in engineering computations very often have elements with many significant figures, which make it impractical to use properties 1 to 5.

c. *Pivotal Condensation*

A more practical way of evaluating determinants was devised by Gauss and is known as *pivotal condensation*. Consider a determinant of the third order,

$$D_3 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

in which $a_1 \neq 0$. If we multiply the second and third rows of D by a_1, the value of D_3 is multiplied by a_1^2 according to rule 2 of Sec. 4·2 d,

$$a_1^2 D_3 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1a_2 & a_1b_2 & a_1c_2 \\ a_1a_3 & a_1b_3 & a_1c_3 \end{vmatrix}$$

while, according to rule 5, the value of this new determinant is unchanged if we subtract from the second and third rows the first row multiplied by a_2 and a_3, respectively,

$$a_1^2 D_3 = \begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & (a_1b_2 - a_2b_1) & (a_1c_2 - a_2c_1) \\ 0 & (a_1b_3 - a_3b_1) & (a_1c_3 - a_3c_1) \end{vmatrix}$$

Expanding by the elements of the first column and writing the elements of the new determinant as determinants of the second order, we finally obtain

$$D_3 = \frac{1}{a_1} \begin{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \\ \\ \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} \end{vmatrix}$$

The computation of D_3 is thus reduced to the computation of a determinant of the second order.

A determinant of order n can be similarly "condensed" to a determinant of order $(n - 1)$ and step by step reduced to a second-order determinant. The element a_{11} used in the condensation is called the *pivot.* When pivotal condensation is applied to an nth-order determinant, we obtain

$$D_n = \frac{1}{a_{11}{}^{n-2}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{31} & a_{3n} \end{vmatrix} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{n1} & a_{n3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix} \tag{4·2·8}$$

which is an $(n - 1)$th-order determinant. If an element a_{ij}, other than a_{11}, is used as pivot, the determinant on the right of Eq. (4·2·8) is multiplied by a factor $(-1)^{i+j}$ but condensation is performed in the same manner.

Examples of pivotal condensation on fourth-order determinants (the pivots are encircled):

$$D_4 = \begin{vmatrix} 2 & 1 & 1 & 1 \\ -1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \end{vmatrix} = \frac{1}{2^{4-2}} \begin{vmatrix} 5 & 3 & 5 \\ 1 & 1 & 3 \\ 2 & 0 & 0 \end{vmatrix} = \frac{1}{2^2}\frac{1}{5^{3-1}} \begin{vmatrix} 2 & 10 \\ -6 & -10 \end{vmatrix}$$

$$= \frac{(-20 + 60)}{20} = 2$$

Using $a_{32} = 1$ as a pivot in D_4, we get, similarly,

$$D_4 = \begin{vmatrix} 2 & 1 & 1 & 1 \\ -1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \end{vmatrix} = \frac{(-1)^{3+2}}{1^{4-2}} \begin{vmatrix} 1 & 0 & -1 \\ -3 & 1 & -2 \\ 0 & -1 & -3 \end{vmatrix}$$

$$= \frac{-1}{1} \frac{1}{1^{3-1}} \begin{vmatrix} -1 & -5 \\ -1 & -3 \end{vmatrix}$$

$$= (-1)(3 - 5) = 2$$

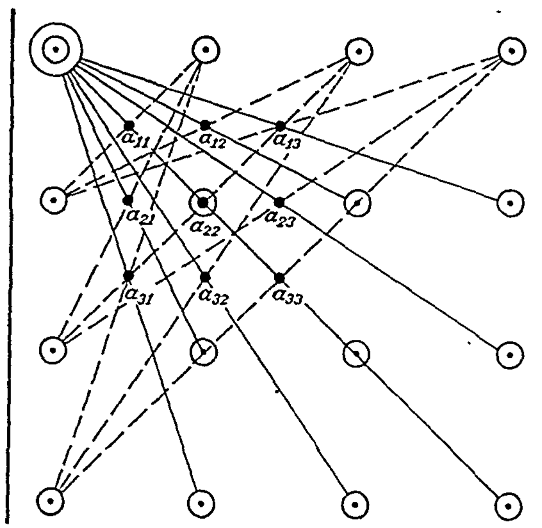


FIG. 4·1.

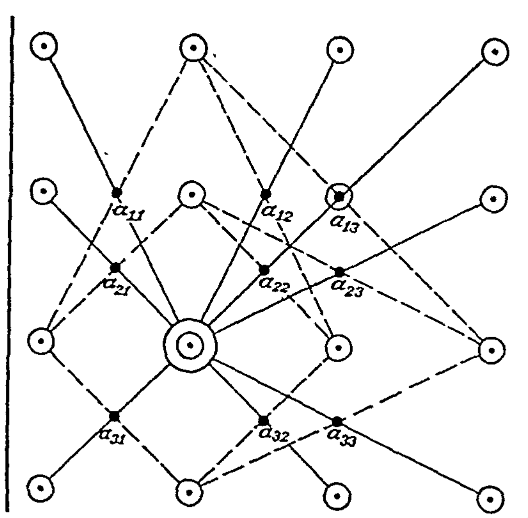


FIG. 4·2.

Pivotal condensation is by far the most practical method for the numerical evaluation of determinants. The element a_{11} is an obviously convenient pivot, and sometimes another element of the determinant is placed in the a_{11} position by interchanging rows and columns before it is used as a pivot.

Figure 4·1 is a diagram for the condensation of fourth-order determinants using a_{11} as a pivot. Each element of the condensed determinant is marked by means of a full circle at the intersection of the lines connecting the elements of the original determinant (white circles) to be multiplied. Full lines indicate positive products, dotted lines negative products. The pivot is marked with two concentric circles.

Figure 4·2 is the same diagram with a_{32} as pivot.

4·3 Gauss's Scheme

The solution by determinants of a system of n equations in n unknowns requires the evaluation of $(n + 1)$ determinants of the nth order. The number of operations necessary for the computation of the $(n + 1)$ determinants by pivotal condensation can be greatly reduced by using a scheme, devised by Gauss, that has been shown to require, in fact, the minimum number of operations among all the known methods of solution of linear equations. The scheme is perfectly general, and its use becomes advisable as soon as the number of equations is greater than three. It is ideally suited to slide-rule computations and prevents the occurrence of errors due to mistaken signs in the computation of determinants.

Gauss's scheme is demonstrated in Table 4·2 for the following system[1] of four equations in four unknowns, presented in tabular form:

	x_1	x_2	x_3	x_4	C
(I)	2	2	4	−2	10
(II)	1	3	2	1	17
(III)	3	1	3	1	18
(IV)	1	3	4	2	27

The equations are labeled by means of Roman numerals; Eq. (I) reads, for example,

$$2x_1 + 2x_2 + 4x_3 - 2x_4 = 10$$

It should be noted that the constants appear on the right-hand side of the equations.

[1] The senior author is indebted to Dr. V. P. Jensen for this numerical example solvable entirely by integers.

In Table 4·2 the first column contains the number of the successive rows of the scheme; the second column the ratios r, to be explained later; the next five columns the coefficients of the unknowns and the constants; the eighth column the so-called "sum-checks," *i.e.*, the sum of all the

TABLE 4·2.—GAUSS'S SCHEME

Number	r	x_1	x_2	x_3	x_4	C	S	Explanations
(1)		$\boxed{2}$	2	4	−2	10	16	(I)
2	$r_2 = 1/2$	1	3	2	1	17	24	(II)
3		−1	−1	−2	1	−5	−8	$-r_2 \times (1)$
(4)		0	$\boxed{2}$	0	2	12	16	(2) + (3)
5	$r_3 = 3/2$	3	1	3	1	18	26	(III)
6		−3	−3	−6	3	−15	−24	$-r_3 \times (1)$
7	$r_3' = -2/2$	0	−2	−3	4	3	2	(5) + (6)
8			2	0	2	12	16	$-r_3' \times (4)$
(9)			0	$\boxed{-3}$	6	15	18	(7) + (8)
10	$r_4 = 1/2$	1	3	4	2	27	37	(IV)
11		−1	−1	−2	1	−5	−8	$-r_4 \times (1)$
12	$r_4' = 2/2$	0	2	2	3	22	29	(10) + (11)
13			−2	0	−2	−12	−16	$-r_4' \times (4)$
14	$r_4'' = 2/-3$		0	2	1	10	13	(12) + (13)
15				−2	4	10	12	$-r_4'' \times (9)$
(16)				0	$\boxed{5}$	20	25	
17	Const. c_i	10	12	15	20			
18	$-x_4a_{i4}$	8	−8	−24	20	$x_4 = 20/5 = 4$		
19	$-x_3a_{i3}$	−12	0	−9	$x_3 = (-9)/(-3) = 3$			
20	$-x_2a_{i2}$	−4	4	$x_2 = 4/2 = 2$				
21	$(i = 1, 4, 9, 16)$	2	$x_1 = 2/2 = 1$					

coefficients and the constant appearing in a given row; the last column the explanation of the operations.

In row (1) appears Eq. (I); row (2) contains Eq. (II). The constant r_2 is obtained by dividing the coefficient of x_1 in row (2) by the coefficient of x_1 in row (1). Row (3) contains Eq. (I) multiplied by $-r_2$.

The sum-check (column S) of row (1) multiplied by $-r_2$ must equal the sum-check of row (3). Whenever the number of equations is larger than three and the coefficients are given with many significant figures, the sum-check must be performed at every row of the scheme.

Row (4) is the sum of rows (2) and (3). It will be noticed that at this point the unknown x_1 has been eliminated between Eqs. (I) and (II).

Row (5) contains Eq. (III). The ratio r_3 is obtained by dividing the coefficient of x_1 in row (5) by the coefficient of x_1 in row (1). Row (6) contains row (1) multiplied by $-r_3$, while row (7) is the sum of rows (5) and (6). Thus x_1 has been eliminated again between Eqs. (I) and (III).

Jumping, for the time being, to rows (10), (11), and (12), we notice that they contain a sequence of operations identical with those of rows (2), (3), and (4), leading to the elimination of x_1 between Eqs. (I) and (IV).

Rows (4), (7), and (12) contain now three equations in the three unknowns x_2, x_3, and x_4. They have actually been obtained by pivotal condensation, with $a_{11} = \boxed{2}$ as a pivot, while the sum-checks guarantee that no obvious errors have been made in the elimination process.

The unknown x_2 is now eliminated between Eqs. (4) and (7) and again between Eqs. (4) and (12), the coefficient of x_2 in Eq. (4), that is, $\boxed{2}$, being used as a pivot. The corresponding ratios r are called r_3' and r_4'. Equations (9) and (14) containing the unknowns x_3 and x_4 only are thus obtained. The unknown x_3 is now eliminated between Eqs. (9) and (14), the coefficient of x_3 in Eq. (9), that is, $\boxed{-3}$, being used as a pivot. Equation (16) thus obtained contains only x_4.

Table 4·3

	x_1	x_2	x_3	x_4	C
(1)	2	2	4	−2	10
(4)		2	0	2	12
(9)			−3	6	15
(16)	.	. .	.	5	20

Equations (1), (4), (9), and (16) (Table 4·3), containing, respectively, 4, 3, 2, and 1 unknown, form what is called a *triangular* system. The triangular equations appear in the rows of Gauss's scheme labeled n^2 ($n = 1, 2, 3, 4$) and are used to obtain the unknowns, working backward

from the last to the first. Thus Eq. (16) reads

$$5x_4 = 20 \tag{b}$$

from which

$$x_4 = 4$$

Equation (9) reads

$$-3x_3 + 6x_4 = 15$$

or

$$-3x_3 = 15 - 6x_4 = 15 - 24 = -9 \tag{c}$$

from which

$$x_3 = 3$$

Equation (4) reads

$$2x_2 + 0x_3 + 2x_4 = 12$$

or

$$2x_2 = 12 - 2x_4 - 0x_3 = 4 \tag{d}$$

from which

$$x_2 = 2$$

And finally Eq. (1) reads

$$2x_1 + 2x_2 + 4x_3 - 2x_4 = 10$$

or

$$2x_1 = 10 + 2x_4 - 4x_3 - 2x_2 = 2 \tag{e}$$

from which

$$x_1 = 1$$

This part of the computation is presented in tabular form in rows (17) to (21) of Gauss's scheme (Table 4·2). Row (17) contains the constants from the triangular equations (1), (4), (9), and (16); row (18) the products of $-x_4$ times the coefficients of x_4 in Eqs. (1), (4), and (9); row (19) the product of $-x_3$ times the coefficients of x_3 in Eqs. (1) and (4); row (20) the product of $-x_2$ times the coefficients of x_2 in Eq. (1). The sums of the figures thus written in the columns x_4, x_3, x_2, and x_1 are the right-hand members of Eqs. (*b*), (*c*), (*d*), and (*e*) above and give the values of the unknowns when divided by the coefficients of x_4 in Eq. (16), of x_3 in Eq. (9), of x_2 in Eq. (4), and of x_1 in Eq. (1), respectively, as shown at the lower part of the scheme.

Once the values of the unknowns are obtained, it is advisable to substitute them in *all* but the first equation to make sure that all the equations are satisfied.

The extension of the scheme to n equations is fairly obvious when one understands the meaning of the operations performed in Gauss's scheme. Some of the figures appearing in the scheme need not actually be written or computed: they have been crossed diagonally by means of dotted lines. The saving of time thus obtained is considerable, but the

beginner should not adopt this short cut until he is entirely familiar with the method.

To simplify the use of the scheme, the pivotal coefficients have been enclosed in a square, and the numerators of the ratios r have been encircled; their denominators are the pivots. The pivots are also the denominators of the fractions giving the unknowns.

It is of the greatest importance to know beforehand how many figures must be carried in the operations of Gauss's scheme to obtain a given number of correct figures in the unknowns, since a large number of figures may be lost in the process of cross multiplication and subtraction. The answer to this question is given by the following approximate error formulas,[1] in which e_m is the maximum error in the roots, e_p the most probable error in the roots, e_c the error in the coefficients, and n the number of equations of the system:

$$e_m \doteq n^6 e_c \qquad (4·3·1)$$

$$e_p \doteq n^4 e_c \qquad (4·3·2)$$

For instance, if the coefficients and constants of a system of six equations are given with three decimal figures and the operations are carried out with three decimal figures, the error e_c is less than 0.001 and by Eq. (4·3·1) the maximum error in the roots may be as high as

$$e_m \leq 6^6 \times 0.001 = 46.7$$

while the most probable error is

$$e_p = 6^4 \times 0.001 = 1.3$$

Conversely, if a most probable error of 0.1 is permissible in the roots, the error in the coefficients must be such that

$$6^4 \times e_c \leq 0.1$$

from which

$$e_c \leq \frac{0.1}{6^4} = 0.00008$$

and at least five decimal figures must be carried through the computations. These examples show that the solution of simultaneous equations is unfortunately very susceptible to errors.

Gauss's scheme can be greatly simplified when the system of equations is *symmetrical*,[2] *i.e.*, when $a_{ij} = a_{ji}$.

[1] These formulas have been kindly communicated to the senior author by Prof. F. J. Murray of Columbia University.

[2] See M. H. Doolittle, *U.S. Coast and Geodetic Survey Rept. for* 1878, pp. 115–120. An ingenious and practical method of solution of simultaneous linear equations, based upon the algebra of a new type of matrices and leading to a triangular system, was given by T. Banachiewicz in 1938 [see *Bull. intern. acad. polon. sci.*, Series A,

4·4 Error Equations

Because of unavoidable errors in the computations, it is very seldom that the roots of a system of n equations can be computed precisely; but when approximate values x_i^0 of the roots have been obtained, it is comparatively easy to improve their accuracy by so-called "error equations," as will be shown for the following system:

	x_1	x_2	x_3	C	
(I)	1	1	1	4.0000	
(II)	2	1	1	5.3333	(a)
(III)	1	2	1	5.3333	

The first seven columns of Table 4·4 show the computations of the approximate values x_1^0, x_2^0, x_3^0 of the roots x_1, x_2, and x_3 of system (a) by Gauss's scheme. If these values x_1^0, x_2^0, and x_3^0 are substituted in the left-hand members of Eqs. (a), some approximate values c_1^0, c_2^0, c_3^0 of the constants will be obtained instead of the correct values $c_1 = 4$, $c_2 = 5.3333$, and $c_3 = 5.3333$.

$$\left.\begin{aligned} x_1^0 + x_2^0 + x_3^0 &= c_1^0 \\ 2x_1^0 + x_2^0 + x_3^0 &= c_2^0 \\ x_1^0 + 2x_2^0 + x_3^0 &= c_3^0 \end{aligned}\right\} \qquad (b)$$

Let us now call δx_i the corrections, which added to the x_i^0 give the correct value of the roots x_i,

$$x_1 = x_1^0 + \delta x_1 \qquad x_2 = x_2^0 + \delta x_2 \qquad x_3 = x_3^0 + \delta x_3 \qquad (4·4·1)$$

and substitute these values for the x_i in the left-hand members of Eqs. (a).

$$\left.\begin{aligned} (x_1^0 + \delta x_1) + (x_2^0 + \delta x_2) + (x_3^0 + \delta x_3) &= c_1 \\ 2(x_1^0 + \delta x_1) + (x_2^0 + \delta x_2) + (x_3^0 + \delta x_3) &= c_2 \\ (x_1^0 + \delta x_1) + 2(x_2^0 + \delta x_2) + (x_3^0 + \delta x_3) &= c_3 \end{aligned}\right\} \qquad (c)$$

Subtracting Eqs. (b) from Eqs. (c) and letting

$$\left.\begin{aligned} e_1 &= c_1 - c_1^0 = c_1 - (x_1^0 + x_2^0 + x_3^0) \\ e_2 &= c_2 - c_2^0 = c_2 - (2x_1^0 + x_2^0 + x_3^0) \\ e_3 &= c_3 - c_3^0 = c_3 - (x_1^0 + 2x_2^0 + x_3^0) \end{aligned}\right\} \qquad (4·4·2)$$

1938, p. 393; also S. Arend, "Voies nouvelles dans le calcul scientifique," *Ciel et terre*, (Brussels), 1941]. Banachiewicz's method is well suited for slide-rule and machine computations.

The same method was independently discovered and tabulated for machine computations by P. D. Crout in 1941 (see *Trans. AIEE*, Vol. 60). Crout's tabular form of solution is recommended whenever the systems to be solved have a large number of unknowns.

TABLE 4-4

Number	r	x_1	x_2	x_3	C	S	$10^3 \times e_i$
(1)		$\boxed{1}$	1	1	4.00	7.00	0
2	$r_2 = 2/1$	2	1	1	5.33	9.33	3.3
3		−2	−2	−2	−8.00	−14.00	0
(4)		0	$\boxed{-1}$	−1	−2.67	−4.67	3.3
5		1	2	1	5.33	9.33	3.3
6	$r_3 = 1/1$	−1	−1	−1	−4.00	−7.00	0
7		0	1	0	1.33	2.33	3.3
8	$r_3' = 1/-1$		−1	−1	−2.67	−4.67	3.3
(9)			0	$\boxed{-1}$	−1.34	−2.34	6.6
10		4.00	−2.67	−1.34	0	3.3	6.6
11		−1.34	1.34	$-1.34/-1$ $=1.34$ $=x_3^0$	6.6	−6.6	$0.0066/-1$ $= -0.0066$ $= \delta x_3$
12		−1.33	−1.33	$-1.33/-1$ $=1.33$ $=x_2^0$	−3.3	−3.3	$-0.0033/-1$ $=0.0033$ $= \delta x_2$
13		+1.33		$1.33/1$ $=1.33$ $=x_1^0$	+3.3		$0.0033/1$ $=0.0033$ $= \delta x_1$

we find that the unknown corrections δx_i satisfy the following system of *error equations*,

$$\left.\begin{aligned} \delta x_1 + \delta x_2 + \delta x_3 &= e_1 \\ 2\delta x_1 + \delta x_2 + \delta x_3 &= e_2 \\ \delta x_1 + 2\delta x_2 + \delta x_3 &= e_3 \end{aligned}\right\} \qquad (4\cdot4\cdot3)$$

where the quantities e_i, defined by Eqs. (4·4·2), are called the *errors due to the roots* x_i^0. The corrections δx_i are thus found to satisfy a system of equations whose coefficients are identical with those of the original system and whose constants are the errors. The e_i are, in general, smaller

than the constants c_i, since the x_i^0 are approximate values of the roots, and the δx_i are therefore smaller than the x_i; the solution of the error equations will thus add one or more significant figures to the roots, even when performed by means of a slide rule.

The error equations cannot, in general, be solved rigorously, and the values of the corrections δx_i will therefore be approximate; but the error-equations procedure can be applied to the error equations themselves to obtain the corrections of the corrections, and the process can be carried on to any degree of accuracy in the roots. In this manner the error-equations process becomes one of successive approximations.

If the initial values x_i^0 are obtained by Gauss's scheme, the evaluation of the corrections δx_i requires the computation of a single additional column, since only the constants of the new system are different from the constants of the original system.

In Table 4·4 the errors corresponding to the initial values $x_1^0 = 1.33$, $x_2^0 = 1.33$, and $x_3^0 = 1.34$ are $e_1 = 0$, $e_2 = 3.3 \times 10^{-3}$, and $e_3 = 3.3 \times 10^{-3}$; column $10^3 \times e_i$ contains these constants and is operated upon as column C was operated upon in the solution of the original system. No sum-check column is used since only a few figures need be computed. The corrections are found to be $\delta x_1 = 0.0033$, $\delta x_2 = 0.0033$, and $\delta x_3 = -0.0066$, and the second approximation of the roots becomes

$$\begin{aligned} x_1' &= 1.33 + 0.0033 = 1.3333 \\ x_2' &= 1.33 + 0.0033 = 1.3333 \\ x_3' &= 1.34 - 0.0066 = 1.3334 \end{aligned}$$

These values are correct within one unit in the fourth decimal figure (better than 1 part in 10,000).

It must be noticed that, since the errors are often differences between figures having many common digits, they must be computed long hand or on a calculating machine in order to obtain enough significant figures.

4·5 Successive Substitutions

When many of the unknowns do not appear in *all* the equations of a system, the method of *successive substitutions* may be usefully employed.

For example, in the system

$$\left.\begin{aligned} x_1 + x_2 \qquad\qquad &= 2 \\ x_1 + 3x_2 + x_3 \qquad &= 5 \\ x_2 + 3x_3 + x_4 &= 5 \\ x_3 + x_4 &= 2 \end{aligned}\right\} \qquad (a)$$

we may derive x_1 from the first equation and x_4 from the last,

$$x_1 = 2 - x_2 \qquad x_4 = 2 - x_3 \qquad (b)$$

and substitute these values in the second and third equations to obtain a system of two equations in x_3 and x_4,

$$\left.\begin{aligned} 2x_2 + x_3 &= 3 \\ x_2 + 2x_3 &= 3 \end{aligned}\right\} \tag{c}$$

From the second of Eqs. (c),

$$x_2 = 3 - 2x_3 \tag{d}$$

and, substituting in the first,

$$2(3 - 2x_3) + x_3 = 3$$

from which

$$x_3 = 1$$

Hence, from Eq. (d),

$$x_2 = 3 - 2 = 1$$

and, from Eq. (b),

$$x_1 = 2 - 1 = 1 \qquad x_4 = 2 - 1 = 1$$

The method of substitutions should be used on only the simplest systems since it easily leads to an accumulation of errors. To eliminate this danger, substitutions should be started, whenever possible, from *both ends* of the system, as shown in the above example.

4·6 Iterative Methods

a. The Gauss-Seidel Method

In the solution of simultaneous equations, determinants have to be abandoned as soon as the order of the system is higher than 5 or 6. Gauss's or Crout's schemes (see Sec. 4·3) may be used to solve up to 30 or 40 equations with the help of a calculating machine, but the amount of labor involved becomes prohibitive in the case of larger systems. Mechanical and electrical devices have also been invented to solve up to 10 equations, while some of the more complicated electronic computers can be used to solve as many as 100 equations. But it is, in general, impossible to solve systems with a very large number of equations by the inexpensive methods of the preceding sections.

An important exception to this rule is represented by those systems in which the coefficient of a different unknown in each equation is much larger than the coefficients of the other unknowns. These systems are called *diagonal*, because the large coefficients are or may be located along the main diagonal of the system, and are encountered in many physical problems. The solution of a system of equations consists actually in making the coefficients of all but one unknown zero in each

equation; hence diagonal systems are, so to speak, well on their way toward a complete solution and can be solved by simple successive approximation or *iterative* methods (from the Latin *iterare*, "to repeat").

Consider the following diagonal system,

	x_1	x_2	x_3	C
(I)	8	1	1	10
(II)	1	10	1	12
(III)	1	1	8	10

(a)

and solve each equation for the unknown with the largest coefficient:

$$\left.\begin{aligned} x_1 &= 1.25 - 0.125x_2 - 0.125x_3 \\ x_2 &= 1.20 - 0.100x_1 - 0.100x_3 \\ x_3 &= 1.25 - 0.125x_1 - 0.125x_2 \end{aligned}\right\} \qquad (b)$$

The system (*a*), written in the form (*b*), is said to be *ready for iteration.*

Gauss's iteration process consists in substituting in the right-hand members of Eqs. (*b*) *any initial values whatsoever* for x_1, x_2, and x_3; in getting from the left-hand members the next approximations x_1', x_2', and x_3'; in substituting these in the right-hand members to obtain the second approximations x_1'', x_2'', and x_3''; and in repeating this sequence of operations until, within the accuracy required by the problem, two successive approximations are identical. It can be proved that, under certain conditions, the numbers thus obtained are the roots of the system. For example, starting with $x_1^0 = x_2^0 = x_3^0 = 0$, we get from Eqs. (*b*)

$$x_1' = 1.25 \qquad x_2' = 1.20 \qquad x_3' = 1.25$$

which, substituted in the right-hand members, give the second approximations

$$x_1'' = 0.944 \qquad x_2'' = 0.950 \qquad x_3'' = 0.944.$$

Table 4·5 shows the results of the first five approximations. The fifth

TABLE 4·5

	1	2	3	4	5
x_1	1.25	0.944	1.013	0.997	1.001
x_2	1.20	0.950	1.011	0.997	1.001
x_3	1.25	0.944	1.013	0.997	1.001

approximation gives the correct value of the roots $x_1 = x_2 = x_3 = 1$ within 1 part in 1000.

The rapidity of convergence of this method can be greatly increased by making use of the *last* computed value of the unknowns at every step, as suggested by Seidel. Thus, starting with $x_2^0 = x_3^0 = 0$, we get, from the first of Eqs. (*b*), $x_1' = 1.25$. We then substitute in the right-hand member of the second equation $x_1' = 1.25$ for x_1 and $x_3^0 = 0$ for x_3 and get $x_2' = 1.075$. Substituting in the third equation $x_1' = 1.25$ and $x_2' = 1.075$, we get $x_3' = 0.959$. Table 4·6 gives the successive approximations of the *Gauss-Seidel method* and shows how much more rapid the convergence of this method is.

TABLE 4·6

	1	2	3
x_1	1.250	0.996	0.999
x_2	1.075	1.005	1.000
x_3	0 959	1.000	1.000

The advantages of the iterative methods are obvious.

1. There is no complicated scheme of operations to be remembered.
2. Errors at any step in the calculations may slow down the convergence but will not influence the final result, since an error is equivalent to a new start with a different initial value.

When rough approximations of the roots are known, as is often the case in engineering problems, these can be used as initial values to save a few steps in the computations. Otherwise, the initial values are taken equal to zero.

Iterative methods cannot be applied to all systems. A sufficient condition for a system to be solvable by iteration is that *in each equation* the sum of the absolute values of all the coefficients except the largest be less than the absolute value of this largest coefficient. *In a system ready for iteration* this condition is equivalent to the following: The sum of the absolute values of the coefficients in the right-hand member of each equation must be less than 1. The smaller the sum, the more rapid the convergence.

In system (*b*) the convergence condition is satisfied abundantly since in the first equation $0.125 + 0.125 \ll 1$, in the second $0.1 + 0.1 \ll 1$, and in the third $0.125 + 0.125 \ll 1$ and hence convergence is very rapid.

The system

$$\begin{aligned} 2x_1 - x_2 - 2x_3 &= -1 \\ x_1 + 2x_2 + x_3 &= 4 \\ x_1 + 3x_2 - x_3 &= 3 \end{aligned}$$

whose roots are $x_1 = x_2 = x_3 = 1$, instead, is *not convergent*, as shown by the successive "approximations" of Table 4·7.

TABLE 4·7

	1	2	3
x_1	−0.50	3.88	5.88
x_2	2.25	1.57	2.73
x_3	3.25	5.59	11.07

When convergence is sufficiently rapid, systems of up to 100 equations can be solved by iteration without too much labor.

b. Converging Increments

When the convergence of the Gauss-Seidel method is not very rapid, another iterative scheme, called the *method of converging increments*, may be used to advantage, although it does not have the convenient property of being uninfluenced by errors in the computations.

The method of converging increments consists in the following steps: (1) computation by the Gauss-Seidel iterative method of the *first* approximation of the roots; (2) computation of the corresponding errors; (3) computation by the Gauss-Seidel iterative method of the *first* approximation of the corrections by means of error equations; (4) computation of the errors of the error equations; (5) computation of the corrections to the corrections by the Gauss-Seidel iterative method; (6) continuation of this process until the last corrections become negligible; (7) summation of the first approximation and of all the successive corrections for each unknown.

The successive corrections of the ith unknown, called *increments*, are indicated by δ_i', δ_i'', δ_i''', etc., and the complete procedure is carried out by means of a simple scheme, which is demonstrated for system (*b*) of Sec. 4·6 *a*, in Table 4·8, to four significant figures in the roots. The first three columns of Table 4·8 contain the three equations (*b*) written *vertically* in tabular form. The coefficients of the same unknown in the three equations appear in the same row, with the signs belonging to them in system (*b*). The constants appear in the fourth row. The process is started by carrying down the constant $c_1 = 1.25$ and encircling it. This gives the value of $x_1' = 1.25$. This value is multiplied by the coefficients of row x_1, and the products are written in columns (II) and (III). The constant $c_2 = 1.200$ and the figure under it, -0.125, are added to give $x_2' = 1.075$, which is encircled. x_2' is multiplied by the coefficients of row x_2, and the products appear in row x_2'. $x_3' = 0.960$, sum of $c_3 = 1.25$ and of the two figures -0.156 and -0.134 appearing under it, is also encircled and multiplied by coefficients of row x_3. The first approximation of the unknowns by the Gauss-Seidel method is thus obtained.

TABLE 4·8

(I)	(II)	(III)	(IV)
1	−0.100	−0.125	x_1
−0.125	1	−0.125	x_2
−0.125	−0.100	1	x_3
1.25	1.200	1.250	c
1.25	−0.125	−0.156	x_1'
−0.134	1.075	−0.134	x_2'
−0.120	−0.096	0.960	x_3'
−0.254	0.025	0.032	δ_1'
0.009	−0.071	0.009	δ_2'
−0.005	−0.004	0.041	δ_3'
0.004	0.000	−0.001	δ_1''
0.001	−0.004	0.001	δ_2''
0	0	0	δ_3''
$x_1 = 1.001$	$x_2 = 1.000$	$x_3 = 1.001$	x_i

The first increment $\delta_1' = -0.254$ is the sum of the two figures under the encircled figures $x_1' = 1.25$. δ_1' is encircled and multiplied by the coefficients of row x_1. $\delta_2' = -0.071$ is the sum of the two figures under $x_2' = 1.075$ and is encircled and multiplied by the coefficients of row x_2. $\delta_3' = 0.041$ is the sum of the two figures under $x_3' = 0.960$ and is encircled and multiplied by the coefficients of row x_3. The successive increments are obtained by repeating the procedure used to obtain the δ_i', and the calculations are stopped in the present case at δ_i'', since the increments δ_i''' would be negligible. The root x_i is the sum of all the encircled figures in the ith column.

Table 4·9 shows the solution of the slowly converging system

$$\begin{aligned} 10x_1 - 5x_2 - 4x_3 &= 1 \\ -3x_1 + 10x_2 - 5x_3 &= 2 \\ -2x_1 - 7x_2 + 10x_3 &= 1 \end{aligned}$$

which, when ready for iteration, becomes

$$\begin{aligned} x_1 &= 0.1 + 0.5x_2 + 0.4x_3 \\ x_2 &= 0.2 + 0.3x_1 + 0.5x_3 \\ x_3 &= 0.1 + 0.2x_1 + 0.7x_2 \end{aligned}$$

It will be noticed that the increments decrease so slowly that after six steps they still affect the second decimal figure of the roots. In this case the roots may be obtained as follows:

In column r appear the ratios of the increment of each root at a given step to the increment of the same root at the preceding step, while column $\bar{r}$ contains the average of these ratios for the three unknowns. It will be noticed that after a few steps these average ratios become very close and then remain practically *constant from step to step.* In Table 4·9 this constant value is 0.7322. If the average ratio $\bar{r}$ remains practically constant after the mth step, we can write approximately

$$\begin{aligned} \delta^{(m+1)} &= \bar{r}\delta^{(m)} \\ \delta^{(m+2)} &= \bar{r}\delta^{(m+1)} = \bar{r}^2\delta^{(m)} \\ \delta^{(m+3)} &= \bar{r}\delta^{(m+2)} = \bar{r}^3\delta^{(m)} \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

The sum of *all* the increments from the mth increment on will then be approximately equal to

$$\delta^{(m)}(1 + \bar{r} + \bar{r}^2 + \cdots + \bar{r}^n + \cdots) = \frac{\delta^{(m)}}{1 - \bar{r}} \tag{4·6·1}$$

since the sum of a geometric series of ratio $\bar{r}$ is equal to $1/(1 - \bar{r})$.

To obtain the value of the unknowns in the case of a slowly converging system, the ratios $\bar{r}$ are computed after a few steps and, as soon as they become practically constant, say at the mth step, the sum of all the increments from the mth on are computed by means of Eq. (4·6·1). The value of the ith unknown is then given by

$$x_i = x_i' + \delta_i' + \delta_i'' + \cdots + \delta_i^{(m-1)} + \frac{\delta_i^{(m)}}{1 - \bar{r}} \tag{4·6·2}$$

Equation (4·6·2) is called the *extrapolation formula* of the converging increments method. In Table 4·9, column x_i gives the successive values of the unknowns; column r the ratios for each unknown; column $\bar{r}$ the average of these ratios; column $1/(1 - \bar{r})$ the sum of the series; column $x_{i,e}$ the value of the unknowns computed by the series summation of Eq. (4·6·2). Comparison of columns x_i and $x_{i,e}$ shows the advantage of the series summation.[1]

[1] A knowledge of matrix theory is needed to prove that the ratios r must approach a constant value.

TABLE 4·9

Steps	(I)	(II)	(III)	x_i	r	$\bar{r}$	$\frac{1}{1-\bar{r}}$	$x_{i,\infty}$
x_1	1.0	0.3	0.2					
x_2	0.5	1.0	0.7					
x_3	0.4	0.5	1.0					
c	0.1	0.2	0.1					
(0)	0.1	0.03	0.02					
	0.1150	0.23	0.161					
	0.1124	0.1405	0.281					
(1)	0.2274	0.0682	0.0455					
	0.1044	0.2087	0.1461					
	0.0766	0.0958	0.1916					
(2)	0.1810	0.0543	0.0362	0.5084	0.7960			1.0125
	0.0750	0.1501	0.1051	0.5888	0.7192	0.7509	4.0144	1.0675
	0.0565	0.0706	0.1413	0.6139	0.7375			1.0499
(3)	0.1315	0.0394	0.0263	0.6399	0.7265			0.9980
	0.0550	0.1100	0.0770	0.6988	0.7328	0.7301	3.7051	0.9948
	0.0413	0.0516	0.1033	0.7172	0.7311			0.9961
(4)	0.0963	0.0289	0.0193	0.7362	0.7323			0.9996
	0.0402	0.0805	0.0564	0.7793	0.7318	0.7323	3.7355	0.9997
	0.0303	0.0378	0.0757	0.7929	0.7328			0.9998
(5)	0.0705	0.0212	0.0141	0.8067	0.7320			0.9995
	0.0295	0.0590	0.0413	0.8383	0.7329	0.7322	3.7341	0.9994
	0.0222	0.0277	0.0554	0.8483	0.7318			0.9999
(6)	0.0517	0.0155	0.0103	0.8584	0.7333			0.9995
	0.0216	0.0432	0.0302	0.8815	0.7322	0.7322	3.7341	0.9996
			0.0405	0.8888	0.7310			0.9998

4·7 Homogeneous Equations

A system of equations is called *homogeneous* when all the constants of the system are equal to zero. A homogeneous system has always a set of roots all equal to zero, since the ith root is given by

$$x_i = \frac{D_i}{D} \tag{4·7·1}$$

and D_i, the determinant obtained from the determinant of the coefficients D by substitution of the constants $c_i = 0$ for the coefficients of x_i, is zero by rule 2 of Art. 4·2 *d*. These roots form the zero-solution, or trivial solution, of the system.

It is obvious from Eq. (4·7·1) that, if a homogeneous system is to have a *nonzero*-solution, the determinant D must be equal to zero, since in this case the roots x_i take the indeterminate form 0/0. For instance, the system

$$\left.\begin{aligned} 3x_1 + 4x_2 - x_3 &= 0 \\ 4x_1 + 8x_2 - 2x_3 &= 0 \\ 5x_1 + 4x_2 - x_3 &= 0 \end{aligned}\right\} \tag{a}$$

may have nonzero-solutions, since its determinant

$$D = \begin{vmatrix} 3 & 4 & -1 \\ 4 & 8 & -2 \\ 5 & 4 & -1 \end{vmatrix} = 0$$

having two proportional columns, is equal to zero by rule 4 of Sec. 4·2 *d*. In this case the system can be solved for the ratios of all the roots to one of them, say x_3. Transposing x_3 to the right-hand member of the equations and considering it as a constant, we may solve the first two equations

$$\begin{aligned} 3x_1 + 4x_2 &= x_3 \\ 4x_1 + 8x_2 &= 2x_3 \end{aligned}$$

for x_1 and x_2 in terms of x_3,

$$x_1 = 0 \qquad x_2 = \frac{x_3}{4}$$

and it is easy to check that these two values satisfy identically the third equation, whatever the value attributed to x_3.

$$5x_1 + 4x_2 - x_3 = 5 \cdot 0 + 4 \cdot \tfrac{1}{4}x_3 - x_3 = 0$$

The ratios of x_1 and x_2 to x_3 are thus found to be

$$\frac{x_1}{x_3} = 0 \qquad \frac{x_2}{x_3} = \frac{1}{4}$$

and it may also be said that the system has an infinite number of solutions, which are found by giving arbitrary values to x_3.

When all the $(n-1)$th-order determinants obtained from the nth-order determinant D by striking one row and one column are also equal to zero, $(n-2)$ roots can be written in terms of the other two roots and will satisfy all the n equations of the system. In general, if all the $(n-m+1)$th-order determinants obtainable from D are equal to zero but at least one $(n-m)$th-order determinant is not zero, $(n-m)$ roots can be written in terms of the other m roots and will satisfy all the n equations of the system.

4·8 Consistency of Equations

The general formula for the evaluation of the roots of a system by determinants,

$$x_i = \frac{D_i}{D}$$

also shows that, if the D_i are different from zero but D equals zero, the roots are not defined. In other words, the condition for a system of nonhomogeneous equations to be solvable uniquely is that $D \neq 0$.

Since systems of equations representing physical phenomena are usually consistent, an inconsistent system will indicate either that the physical problem has not been properly formulated in mathematical terms or that numerical errors have been made in the computations.

Problems

1. Solve by elimination the following systems of equations for x and y:

(a) $\begin{cases} 3x + ay = 5 \\ 2x - by = 6 \end{cases}$ (b) $\begin{cases} \dfrac{x}{7} - \dfrac{3y}{14} = -c \\ \dfrac{x}{2} + \dfrac{7y}{6} = 8d \end{cases}$

(c) $\begin{cases} \dfrac{4}{x} - \dfrac{3}{y} = 1 \\ \dfrac{7}{x} + \dfrac{4}{y} = 11 \end{cases}$ (d) $\begin{cases} 4x^2 + 7y^2 = 23 \\ 3x^2 + 5y^2 = 17 \end{cases}$

(e) $\begin{cases} 3x - 4y = 5 \\ -6x + 8y = -10 \end{cases}$ (f) $\begin{cases} 1.17x + 2.42y = 6.01 \\ 3.12x - 2.75y = -2.38 \end{cases}$

2. Solve Prob. 1 by determinants.

3. Solve by determinants the following systems, using the Laplacian expansion to evaluate the determinants:

(a) $\begin{cases} x + y - z = 0 \\ 4x + 3y - z = 7 \\ 6x - 5y - z = -7 \end{cases}$ (b) $\begin{cases} 2x + 5y + 3z = 7 \\ 3x + 2y - 4z = -1 \\ 5x + 9y - 7z = -8 \end{cases}$

(c) $\begin{cases} 4x - 5y - 6z = 33 \\ x - y + z = 3 \\ 9x + z = 34 \end{cases}$

4. By using the properties of determinants, reduce

$$\begin{vmatrix} 2 & -2 & 4 \\ -2 & -4 & -1 \\ 3 & -3 & 2 \end{vmatrix}$$

to the form

$$\begin{vmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{vmatrix}$$

5. Prove that

$$\begin{vmatrix} 1 + a^2 + b^2 & 2ab & -2b \\ 2ab & 1 + b^2 - a^2 & 2a \\ 2b & -2a & 1 - a^2 - b^2 \end{vmatrix} = (1 + a^2 + b^2)^3$$

6. Prove the following identity:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} x & y \\ u & v \end{vmatrix} = \begin{vmatrix} ax + bu & ay + bv \\ cx + du & cy + dv \end{vmatrix}$$

7. Check that

$$\begin{vmatrix} a + x & c + u \\ b + y & d + v \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} + \begin{vmatrix} x & u \\ y & v \end{vmatrix} + \begin{vmatrix} a & u \\ b & v \end{vmatrix} + \begin{vmatrix} x & c \\ y & d \end{vmatrix}$$

8. If (a_1,b_1), (a_2,b_2), (a_3,b_3) are the coordinates of the vertices of a triangle, show that its area A may be expressed by

$$A = \frac{1}{2} \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

9. Establish from fundamental considerations the following identity:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_1 + xb_1 & b_1 & c_1 \\ a_2 + xb_2 & b_2 & c_2 \\ a_3 + xb_3 & b_3 & c_3 \end{vmatrix}$$

10. Derive the following formula for the derivative of a determinant, in which f_i,g_i,h_i $(i = 1, 2, 3)$ are functions of x:

$$\frac{d}{dx} \begin{vmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{vmatrix} = \begin{vmatrix} f_1' & g_1 & h_1 \\ f_2' & g_2 & h_2 \\ f_3' & g_3 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & g_1' & h_1 \\ f_2 & g_2' & h_2 \\ f_3 & g_3' & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & g_1 & h_1' \\ f_2 & g_2 & h_2' \\ f_3 & g_3 & h_3' \end{vmatrix}$$

Primes denote differentiation with respect to x.

11. Establish the identity

$$\begin{vmatrix} A & 1 & 1 & 1 \\ 1 & A & 1 & 1 \\ 1 & 1 & A & 1 \\ 1 & 1 & 1 & A \end{vmatrix} = (A - 1)^3(A + 3)$$

12. Evaluate the following determinants by pivotal condensation, using as pivot in each example (1) a_{11}, (2) a_{22}, and (3) a_{32}:

$$(a)\ \begin{vmatrix} 1 & 3 & -2 & 4 \\ -1 & -4 & 3 & 7 \\ 2 & 1 & 7 & 0 \\ 1 & -3 & -1 & 1 \end{vmatrix} \qquad (b)\ \begin{vmatrix} 1.31 & 6.15 & 3.14 \\ 7.40 & -2.10 & 6.30 \\ 5.10 & 6.42 & 1.00 \end{vmatrix}$$

$$(c)\ \begin{vmatrix} 2.1 & 3.1 & -4.2 & 4.1 \\ -1.2 & 1.4 & 1.7 & -1.6 \\ 7.2 & -6.9 & 7.0 & 6.8 \\ 8.1 & 2.2 & 1.1 & 0.5 \end{vmatrix}$$

13. Evaluate by determinants, using pivotal condensation, the roots of the following system:

x_1	x_2	x_3	x_4	c
1	2	2	1	0
2	−4	6	−4	−4
0	1	1	0	0
1	0	0	1	0

14. Solve by Gauss's scheme to 2 significant figures the following systems:

(a)

x_1	x_2	x_3	x_4	c
1.5	7.0	3.0	0	5.5
2.4	3.0	−1.4	−1.7	3.4
6.3	0	−4.0	2.1	14.5
0	4.0	3.0	−2.0	−3.0

(b)

x_1	x_2	x_3	x_4	c
2.7	−3.1	4.2	−6.1	0.5
3.4	7.2	−6.3	4.5	2.3
−5.1	−6.4	3.2	7.3	14.0
5.0	4.3	−3.6	−2.8	−3.5

(c)

x_1	x_2	x_3	x_4	x_5	c
2.6	3.1	0.65	12.6	3.9	9.43
−0.4	1.2	3.4	−7.2	0.9	23.03
3.1	2.2	−1.6	0.6	1.2	−0.229
−1.2	3.2	−2.4	0.9	0.7	−10.89
0.7	−0.6	2.3	0.8	−0.5	+3.37

15. Obtain one more significant figure in the roots of Prob. 14c by means of error equations

16. Solve the following system for x,y,z, and t in terms of a, sin α, cos α, sin 3α, and cos 3α:

$$\begin{aligned} x\cos\alpha + y\cos 3\alpha + z\sin\alpha + t\sin 3\alpha &= 0 \\ z + 3t &= 0 \\ 2x\sin\alpha + 6y\sin 3\alpha - 2z\cos\alpha - 6t\cos 3\alpha &= 0 \\ 6x + 6y &= -a \end{aligned}$$

17. Solve by successive substitutions the following systems, starting from both ends in systems (a) and (b):

$$(a)\ \begin{cases} x_1 - 3x_2 = -2 \\ 2x_2 + 4x_3 = 8 \\ x_1 + 6x_3 = 10 \end{cases} \qquad (b)\ \begin{cases} x_1 - 4x_2 + x_3 = -4 \\ x_1 + x_3 = 4 \\ 2x_1 + 2x_2 + x_4 = 10 \\ x_3 - x_4 = -1 \end{cases}$$

$$(c)\ \begin{cases} x_1 + x_2 = 3 \\ x_2 + x_3 = 5 \\ x_3 + x_4 = 7 \\ x_4 + x_5 = 9 \\ x_1 + x_2 + x_3 + x_4 + x_5 = 15 \end{cases}$$

18. Solve by the Gauss-Seidel iteration method the following systems:

$$(a)\ \begin{cases} 6x_1 - 2x_2 - x_3 = 9 \\ x_1 + 7x_2 + x_3 = 10 \\ x_1 - x_2 + 8x_3 = 9 \end{cases} \qquad (b)\ \begin{cases} 12.4x_1 - x_2 - 2x_3 = 27.8 \\ 2.1x_1 + 17.5x_2 - 3.1x_3 = -10.2 \\ 1.2x_1 - 0.5x_2 + 11.1x_3 = -8.2 \end{cases}$$

$$(c)\ \begin{cases} 10x_1 + x_2 + x_3 + x_4 = 14 \\ x_1 - 10x_2 - x_3 - x_4 = -11 \\ -x_1 + x_2 + 10x_3 + x_4 = 11 \\ x_1 + x_2 - x_3 + 10x_4 = 11 \end{cases}$$

19. Solve Prob. 18 by the method of converging increments, using extrapolation for system (a).

20. Compute the roots of the following system to 3 significant figures by converging increments, using extrapolation:

$$\begin{aligned} 5x_1 - x_2 + x_3 &= 5.75 \\ x_1 + 4x_2 + x_3 &= 9.45 \\ x_1 + 3x_2 + 6x_3 &= 15.00 \end{aligned}$$

21. Check for consistency the following systems:

$$(a)\ \begin{cases} -4\ x_1 + 3\ x_2 + 2x_3 = 6 \\ 2.5x_1 - 7\ x_2 + 5x_3 = -4 \\ 2\ x_1 - 7.5x_2 - 3x_3 = 7 \end{cases} \qquad (b)\ \begin{cases} x_1 + x_2 + 3x_3 = 5 \\ 2x_1 - x_2 + 3x_3 = 6 \\ -x_1 + 3x_2 + x_3 = 5 \end{cases}$$

22. What values must λ have in order that the following system may have nonzero roots?

$$\begin{aligned} (3 - \lambda)x_1 + 2x_2 &= 0 \\ x_1 + (1 - \lambda)x_2 &= 0 \end{aligned}$$

23. Find the ratios of x and y to z such that x, y, and z be roots of the following system:

$$\begin{aligned} x + y + 4z &= 0 \\ 2x - y - z &= 0 \\ 3x + 2y + 9z &= 0 \end{aligned}$$

24. Give the values of x and y as functions of z,t that satisfy the following system:

$$\begin{aligned} x + y + z + t &= 0 \\ x + 3y + 2z + 2t &= 0 \\ 5x + 3y + 4z + 4t &= 0 \\ 7x + 9y + 8z + 8t &= 0 \end{aligned}$$

25. Determine x such that the following determinant has a minimum value, and compute this minimum value:

$$\begin{vmatrix} x & 2x & 3 \\ -4 & x & -x \\ x & 1 & -x \end{vmatrix}$$

26. In a surveying problem a quadrilateral is obtained such that the sum of the 2 smaller angles is 54 deg and the difference of the 2 larger angles is 14 deg. The largest angle is 4 times one of the smaller. What are the 4 angles?

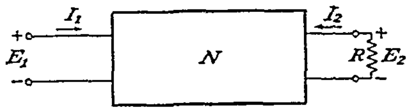
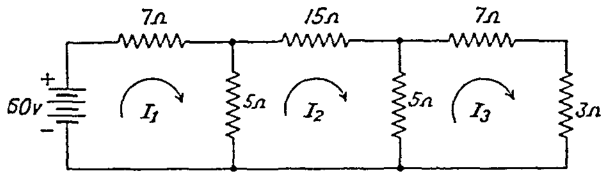


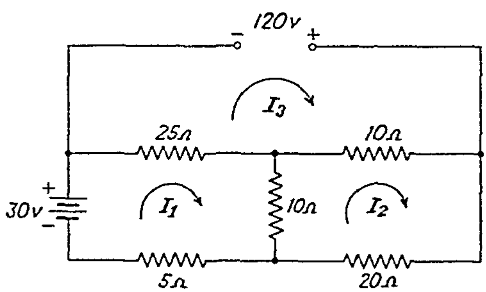
Fig. 4·3.

27. The equilibrium equations of a 4-terminal network (see Fig. 4·3) are

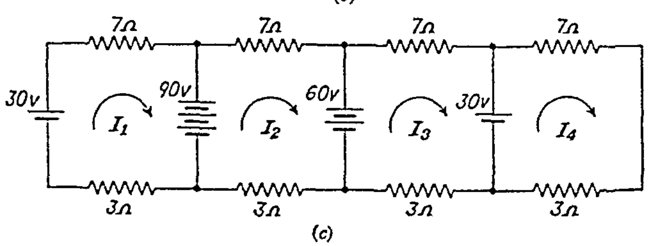
$$E_1 = z_{11}I_1 + z_{12}I_2$$
$$E_2 = z_{12}I_1 + z_{22}I_2$$



(a)



(b)



(c)

Fig. 4·4.

where E_1 and E_2 are the terminal voltages, I_1 and I_2 the currents, and z_{11}, z_{12}, and z_{22} are impedances depending upon the network parameters. If $E_2 = RI_2$, determine the transfer impedance Z_{12} of the network in terms of E_1, E_2, z_{11}, z_{12}, and z_{22}.

Hint: The transfer impedance is defined as E_2/I_1.

28. Find the loop currents in the networks shown in Fig. 4·4.

29. Determine the equation of the conic passing through the 5 points (0, 1), $(1, -8/9)$, $(-4, 57)$, $(2, 3/11)$, and $(-1, 0)$. *Hint:* The general equation of the conic is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

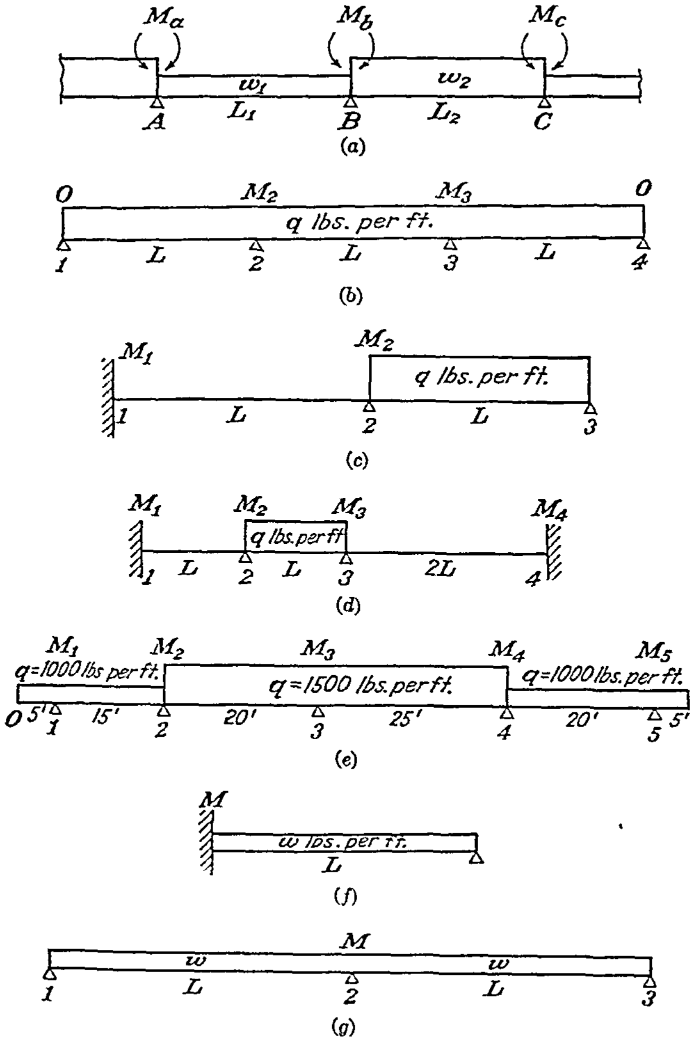


FIG. 4·5.

30. Find the moments at the supports of the continuous beams of Fig. 4.5*b*, *c*, *d*, *e*, *f*, and *g* by the theorem of 3 moments,

$$M_A L_1 + 2M_B(L_1 + L_2) + M_C L_2 + \frac{w_1 L_1^3}{4} + \frac{w_2 L_2^3}{4} = 0$$

where w_1 and w_2 are the loads per unit of length on the spans of length L_1 and L_2, respectively (Fig. 4.5*a*).

Note: All beams have constant flexural rigidity EI, where E is the modulus of elasticity in pounds per square inch and I the transverse moment of inertia in in.^{4}

Hint: The moment at a built-in end is found by prolonging the beam symmetrically on the other side of the built-in end so that the prolonged beam has zero slope at the support. For example, the single-span beam of Fig. 4·5*f* has the same deflections as the 2-span beam of Fig. 4·5*g*.

31. The following table gives the temperature T of sea water in degrees Centigrade at various depths d in feet, as recorded by experimental investigators:

i	1	2	3	4	5	6	7
d_i	0	10	20	30	40	50	60
T_i	30	25	23	19	13	9	6

Determine by least squares the best straight-line fit to this table. *Hint:* Letting the equation of the straight line be $T = Ad + B$, the constants A and B are such as to minimize the sum of the square of the "residuals,"

$$S = \sum_{i=1}^{7} (Ad_i + B - T_i)^2$$

Hence A and B satisfy the 2 equations

$$\frac{\partial S}{\partial A} = 0 \qquad \frac{\partial S}{\partial B} = 0$$

CHAPTER V

ELEMENTARY FUNCTIONS AND POWER SERIES

5·1 Elementary Functions

Algebraic, trigonometric, inverse trigonometric, logarithmic, exponential, hyperbolic, and inverse hyperbolic functions are called *elementary functions* because they are often encountered in elementary problems and have been extensively tabulated. Their fundamental properties will be briefly reviewed in this chapter.

5·2 Algebraic Functions

a. A beam of length L, moment of inertia I, and modulus of elasticity E, loaded uniformly with a load of q lb per ft, is elastically connected to two columns and cannot be considered as either simply supported or built in at the ends. The actual deflections of the beam will certainly be smaller than the deflections of the simply supported beam and larger than those of the built-in beam. Being unable to define the actual support conditions, we shall consider these two extreme cases.

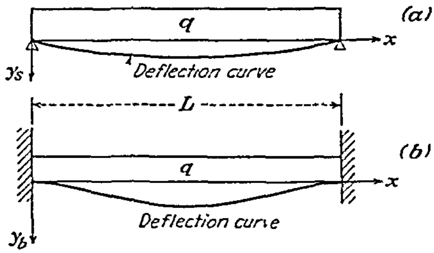


FIG. 5·1.

The simply supported and built-in beam deflections are given, respectively, by[1]

$$y_s = \frac{qL^4}{24EI}(z^4 - 2z^3 + z)$$

$$y_b = \frac{qL^4}{24EI}(z^4 - 2z^3 + z^2)$$

where

$$z = \frac{x}{L}$$

and x is the distance, from the left end of the beam, of the section whose deflection is y (Fig. 5·1).

[1] See, for instance, S. Timoshenko, "Strength of Materials," Chaps. V and VI, Van Nostrand Company, Inc., New York, 1930.

y_s and y_b are two examples of the algebraic functions called real *polynomials, i.e.*, of linear combinations with real constants of integral powers of the variable. Polynomials are single-valued, continuous functions with any number of continuous derivatives. The successive derivatives of y_s, for instance, have known physical significance; the first derivative

$$\frac{dy_s}{dz} = \frac{qL^4}{24EI}(4z^3 - 6z^2 + 1)$$

is proportional to the rotation θ of the sections of the beam; the second

$$\frac{d^2y_s}{dz^2} = \frac{qL^4}{2EI}(z^2 - z)$$

is proportional to the bending moment; the third

$$\frac{d^3y_s}{dz^3} = \frac{qL^4}{2EI}(2z - 1)$$

is proportional to the shear; the fourth

$$\frac{d^4y_s}{dz^4} = \frac{qL^4}{EI}$$

is proportional to the load.

The simplest way of evaluating polynomials is by synthetic division, as explained in Sec. 3·4 *d*. For instance, the value of y_s at the middle of the simply supported beam ($z = 0.5$) is found as follows:

$$\begin{array}{r|rrrrr} & 1 & -2.00 & 0.00 & 1.000 & 0.0000 \\ 0.5 & & 0.50 & -0.75 & -0.375 & 0.3125 \\ \hline & 1 & -1.50 & -0.75 & 0.625 & \boxed{0.3125} \end{array}$$

$$y_s(0.5) = 0.3125\,\frac{qL^4}{24EI}$$

while the deflection at the middle of the built-in beam is given by

$$\begin{array}{r|rrrrr} & 1 & -2.0 & 1.00 & 0.000 & 0.0000 \\ 0.5 & & 0.5 & -0.75 & 0.125 & 0.0625 \\ \hline & 1 & -1.5 & 0.25 & 0.125 & \boxed{0.0625} \end{array}$$

$$y_b(0.5) = 0.0625\,\frac{qL^4}{24EI}$$

A polynomial containing only even powers of the variable x remains unchanged if x is changed into $-x$.

$$2(-x)^4 + 4(-x)^2 + 2 = 2x^4 + 4x^2 + 2$$

By analogy with even polynomials any function for which

$$f(-x) = f(x) \tag{5·2·1}$$

is called *even.*

Similarly, a polynomial containing only odd powers of x is changed in sign if x is changed into $-x$,

$$2(-x)^5 + 3(-x)^3 + (-x) = -(2x^5 + 3x^3 + x)$$

and any function for which

$$f(-x) = -f(x) \tag{5·2·2}$$

is called *odd.*

In order to evaluate the relative difference between y_s and y_b we may consider their ratio

$$r(z) = \frac{y_s}{y_b} = \frac{z^4 - 2z^3 + z}{z^4 - 2z^3 + z^2}$$

$r(z)$ is a rational function called an *algebraic fraction* and may be evaluated by long division as follows:

$$\begin{array}{ccccc|ccccc}
1 & -2 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 0 \\ \cline{6-10}
1 & -2 & 1 & 0 & 0 & 1 & & & & \\ \cline{1-5}
0 & 0 & -1 & 1 & 0 & & & & &
\end{array}$$

from which

$$\begin{aligned}
r(z) &= 1 + \frac{-z^2 + z}{z^4 - 2z^3 + z^2} = 1 + \frac{-z + 1}{z^3 - 2z^2 + z} \\
&= 1 + \frac{1}{z(1 - z)} = \frac{-z^2 + z + 1}{z(1 - z)}
\end{aligned}$$

$r(z)$ is minimum at $z = 0.5$, where $r(0.5) = 5$, and approaches infinity as z approaches 0 and as z approaches 1. The values of the variable at which a function becomes infinite are called the *poles* of the function. The poles of an algebraic fraction are those roots of the equation, obtained by setting its denominator equal to zero, which are not roots of its numerator. The poles of $r(z)$ are the roots of $z(1 - z) = 0$, that is, $z = 0$ and $z = 1$, because $z = 0$ and $z = 1$ are not roots of

$$z^2 - z - 1 = 0$$

b. An operation that must often be performed in engineering computations is the splitting of an algebraic fraction into *partial fractions, i.e.,* into fractions, whose denominators are binomials. For instance, the fraction $1/z(z - 1)$ may be split into two partial fractions

$$\frac{1}{z(z - 1)} = \frac{a}{z} + \frac{b}{z - 1} = \frac{a}{z - 0} + \frac{b}{z - 1}$$

where a and b are constants.

To evaluate a, multiply both sides of this equation by z, and set $z = 0$.

$$\left.\frac{1}{z-1}\right]_{z=0} = a + \left.\frac{bz}{z-1}\right]_{z=0} = a$$

from which

$$a = -1$$

Similarly, multiplying both sides of the equation by $z - 1$ and setting $z = 1$, we obtain

$$\left.\frac{1}{z}\right]_{z=1} = \left.\frac{a}{z}(z-1) + b\right]_{z=1}$$

from which

$$b = 1$$

Hence

$$\frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$

In general, when the denominator of the fraction $A(z)/B(z)$, in which the degree of A is less than the degree of B,[1] has n separate real roots z_i, the fraction can be written as

$$\frac{A(z)}{B(z)} = \sum_{i=1}^{n} \frac{C_i}{z - z_i} \tag{5·2·3}$$

where the constants C_i are given by

$$C_i = \left.\frac{(z - z_i)A(z)}{B(z)}\right]_{z=z_i} \tag{5·2·4}$$

The same formulas can be used when the denominator $B(z)$ has separate couples of complex conjugate roots, but in this case the constants C_i are complex. In order to avoid complex constants it is customary to combine in a single fraction, with linear numerator and quadratic denominator, the two partial fractions corresponding to a couple of conjugate complex roots. Thus the fraction

$$\frac{1}{(z-1)(z^2+z+1)}$$

may be split into the two fractions

$$\frac{1}{(z-1)(z^2+z+1)} = \frac{a}{z-1} + \frac{bz+c}{z^2+z+1} \tag{a}$$

[1] When the degree of A is greater than the degree of B, A is first divided by B.

where a, b, and c are real constants. The value of a is given, as before, by

$$a = \left.\frac{1}{z^2 + z + 1}\right]_{z=1} = \frac{1}{3}$$

To obtain the values of b and c, multiply by $z^2 + z + 1$ both sides of Eq. (a),

$$\frac{1}{z-1} = \frac{a}{z-1}(z^2 + z + 1) + bz + c$$

and set $z^2 + z + 1 = 0$, that is,

$$z^2 = -z - 1 \tag{b}$$

Thus

$$\frac{1}{z-1} = bz + c$$

and, multiplying both sides by $(z - 1)$,

$$1 = bz^2 + (c - b)z - c$$

or, again making use of (b),

$$1 = b(-z - 1) + (c - b)z - c = (c - 2b)z - (b + c)$$

For this equation to hold for all values of z, the coefficients of the various powers of z must be separately equal, and hence

$$c - 2b = 0 \qquad b + c = -1$$

from which

$$b = -\tfrac{1}{3} \qquad c = -\tfrac{2}{3}$$

and

$$\frac{1}{(z-1)(z^2 + z + 1)} = \frac{1}{3}\left(\frac{1}{z-1} - \frac{z+2}{z^2 + z + 1}\right)$$

When some of the n roots of the denominator $B(z)$ are repeated, the number of partial fractions is still n, because to each root repeated k times there correspond k partial fractions, as shown in the following example:

$$\frac{1}{(z-1)(z-2)^2} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2} \tag{c}$$

The root $z = 2$, which is repeated twice, gives rise to two partial fractions with denominators $z - 2$ and $(z - 2)^2$. When the root z_i is repeated k times, the corresponding partial fractions have denominators equal to $z - z_i, (z - z_i)^2, \ldots, (z - z_i)^k$. The constant A is obtained as before, by multiplying both sides of Eq. (c) by $z - 1$,

$$\frac{1}{(z-2)^2} = A + B\frac{z-1}{z-2} + C\frac{z-1}{(z-2)^2}$$

and setting $z = 1$,

$$A = \frac{1}{(1-2)^2} = 1$$

Similarly, to obtain C, multiply both sides of Eq. (c) by $(z-2)^2$,

$$\frac{1}{z-1} = A\,\frac{(z-2)^2}{z-1} + B(z-2) + C \qquad (d)$$

and set $z = 2$,

$$C = \frac{1}{2-1} = 1$$

To obtain B, differentiate both sides of Eq. (d) with respect to z,

$$\frac{-1}{(z-1)^2} = A\,\frac{(z-1)\cdot 2\cdot(z-2) - (z-2)^2}{(z-1)^2} + B$$

and set $z = 2$,

$$\frac{-1}{(2-1)^2} = A\cdot 0 + B$$

Therefore

$$B = -1$$

and

$$\frac{1}{(z-1)(z-2)^2} = \frac{1}{z-1} - \frac{1}{z-2} + \frac{1}{(z-2)^2}$$

When a root is repeated k times, $(k-1)$ successive differentiations will be needed to obtain all the constants.

5·3 Trigonometric Functions

The three fundamental *trigonometric*, or *circular, functions* (sine, cosine, and tangent) are met in a great number of engineering problems. They are single-valued; the sine and the tangent are odd functions, and the cosine *is an even function.*

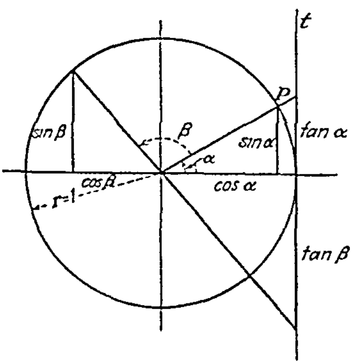


FIG. 5·2.

Figure 5·2 shows the segments whose lengths measure the sine, cosine, and tangent on a circle of radius equal to unity (*unit circle*). It is easy to obtain by the unit circle the relationships between the values of the trigonometric functions for angles in the second, third, and fourth quadrant and the values of the same functions for angles in the first quadrant (Table 5·1).

TABLE 5·1

$\sin\left(\alpha + \frac{\pi}{2}\right) = \cos\alpha$	$\cos\left(\alpha + \frac{\pi}{2}\right) = -\sin\alpha$	$\tan\left(\alpha + \frac{\pi}{2}\right) = -\frac{1}{\tan\alpha}$
$\sin(\alpha + \pi) = -\sin\alpha$	$\cos(\alpha + \pi) = -\cos\alpha$	$\tan(\alpha + \pi) = \tan\alpha$
$\sin\left(\alpha + \frac{3\pi}{2}\right) = -\cos\alpha$	$\cos\left(\alpha + \frac{3\pi}{2}\right) = \sin\alpha$	$\tan\left(\alpha + \frac{3\pi}{2}\right) = -\frac{1}{\tan\alpha}$

The tangent has poles (becomes infinite) at the odd multiples of $\pi/2$, approaching plus infinity when the multiples of $\pi/2$ are approached from below and minus infinity when the multiples of $\pi/2$ are approached from above.

In the calculus the variable of the trigonometric functions is always measured in radians, for the reason explained in Sec. 1·7 *b*3.

Table 5·2 gives the trigonometric identities most commonly used in the derivations of engineering problems.

TABLE 5·2

(*a*) $\sin^2\alpha + \cos^2\alpha = 1$

(*b*) $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

(*c*) $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

(*d*) $\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

(*e*) $\sin 2\alpha = 2\sin\alpha\cos\alpha$

(*f*) $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

(*g*) $\tan 2\alpha = \dfrac{2\tan\alpha}{1 - \tan^2\alpha}$

(*h*) $\sin\frac{1}{2}\alpha = \sqrt{\dfrac{1 - \cos\alpha}{2}}$

(*i*) $\cos\frac{1}{2}\alpha = \sqrt{\dfrac{1 + \cos\alpha}{2}}$

(*j*) $\tan\frac{1}{2}\alpha = \sqrt{\dfrac{1 - \cos\alpha}{1 + \cos\alpha}}$

(*k*) $\sin\alpha \pm \sin\beta = 2\sin\frac{1}{2}(\alpha \pm \beta)\cos\frac{1}{2}(\alpha \pm \beta)$

(*l*) $\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha + \beta)\cos\frac{1}{2}(\alpha - \beta)$

(*m*) $\cos\alpha - \cos\beta = -2\sin\frac{1}{2}(\alpha + \beta)\sin\frac{1}{2}(\alpha - \beta)$

(*n*) $\tan\alpha \pm \tan\beta = \dfrac{\sin(\alpha \pm \beta)}{\cos\alpha\cos\beta}$

(*o*) $\sin\alpha\sin\beta = \frac{1}{2}\cos(\alpha - \beta) - \frac{1}{2}\cos(\alpha + \beta)$

(*p*) $\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha + \beta) + \frac{1}{2}\cos(\alpha - \beta)$

(*q*) $\sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha + \beta) + \frac{1}{2}\sin(\alpha - \beta)$

5·4 Inverse Trigonometric Functions

The bottom and top of the screen of a motion-picture theater are located *a* and *b* ft, respectively, above the horizontal floor of the house. How far from the screen should one sit to see the screen at the largest possible angle?

From Fig. 5·3 the angle θ, at which the screen is seen by a spectator x ft from the screen, is given by

$$\theta = \alpha_2 - \alpha_1$$

where the angles α_2 and α_1, at which the spectator sees the bottom and the top of the screen, are defined by their tangents

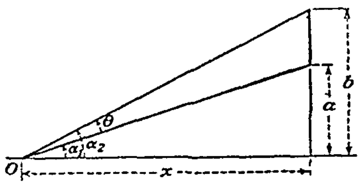


FIG. 5·3.

$$\tan \alpha_1 = \frac{a}{x} \qquad \tan \alpha_2 = \frac{b}{x}$$

In the following computations it is convenient to consider the angles α_1 and α_2 as functions of the variable x by means of the inverse trigonometric function *arctangent*

$$\alpha_1 = \arctan \frac{a}{x} \qquad \alpha_2 = \arctan \frac{b}{x}$$

and to write the angle θ as

$$\theta = \arctan \frac{b}{x} - \arctan \frac{a}{x}$$

To find the value of x for which θ becomes maximum, we set equal to *zero the derivative of θ with respect to x.*

$$\frac{d\theta}{dx} = \frac{d}{dx}\left(\arctan \frac{b}{x}\right) - \frac{d}{dx}\left(\arctan \frac{a}{x}\right) = 0$$

The derivative of the inverse function $z = \arctan y$ is obtained by the rule of Sec. 1·7 c3.

$$y = \tan z \qquad z = \arctan y$$

$$\frac{dy}{dz} = \frac{1}{\cos^2 z} \qquad \frac{dz}{dy} = \cos^2 z = \frac{1}{1 + \tan^2 z} = \frac{1}{1 + y^2}$$

Hence

$$\begin{aligned} \frac{d\theta}{dx} &= \frac{1}{1 + (b/x)^2}\left(-\frac{b}{x^2}\right) - \frac{1}{1 + (a/x)^2}\left(-\frac{a}{x^2}\right) \\ &= -\frac{b}{x^2 + b^2} + \frac{a}{x^2 + a^2} \\ &= \frac{(a - b)(x^2 - ab)}{(x^2 + b^2)(x^2 + a^2)} = 0 \end{aligned}$$

and

$$x^2 - ab = 0$$

or

$$x = \sqrt{ab}$$

For instance, with $a = 12$ ft and $b = 24$ ft,

$$x = 12\sqrt{2} = 16.97 \text{ ft}$$

$$\tan \alpha_2 = \frac{24}{16.97} \qquad \alpha_2 = 51°45'$$

$$\tan \alpha_1 = \frac{12}{16.97} \qquad \alpha_1 = 35°15'$$
$$\theta_{\max} = 16°30'$$

That this value of θ is actually a maximum may be seen, without investigation of the sign of $d^2\theta/dx^2$, by noticing that θ is always positive and approaches zero both as x approaches zero and as x approaches infinity.

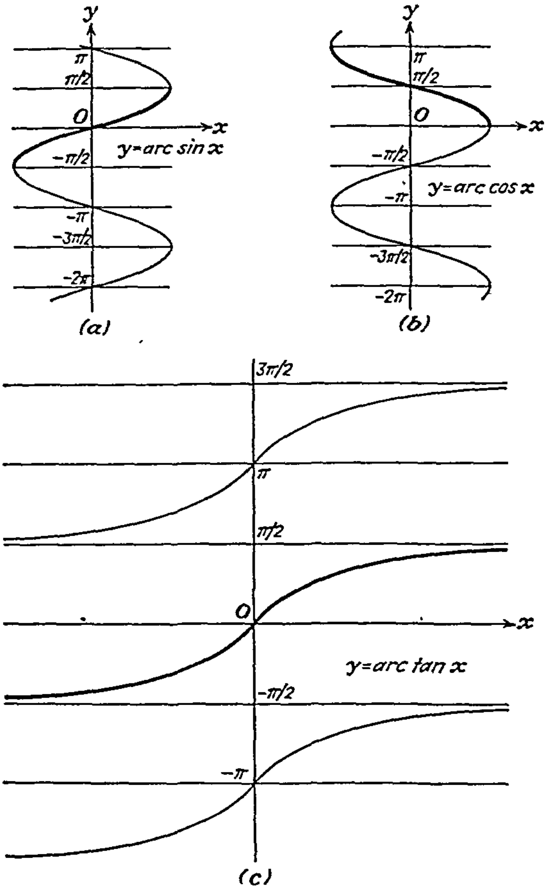


FIG. 5·4.

The inverse functions of the sine and cosine are called *arcsine* and *arccosine*. The inverse trigonometric functions are sometimes written by means of the symbols

$$\sin^{-1} x \qquad \cos^{-1} x \qquad \tan^{-1} x$$

and are *not* single-valued, since to a given value of the sine, cosine, or tangent there corresponds an infinity of values of the arc. The graphs of the arcsine, arccosine, and arctangent appear in Fig. 5·4*a*, *b*, and *c*.

The inverse trigonometric functions become single-valued if the interval of definition of arcsin x and arctan x is limited between $-\pi/2$ and $\pi/2$ and the interval of definition of arccos x is limited between 0 and π. The corresponding values of the inverse functions are called their *principal values* and are indicated by a heavy line in Fig. 5·4.

5·5 The Logarithmic Function

A magazine offers two types of subscription rates: a life subscription for \$50 and a yearly subscription for \$5. If bank deposits bring 2 per cent interest, after how many years does the first form of subscription become more convenient than the second?

The sum of \$50 in a bank account becomes

$$50(1 + 0.02) = \$51$$

at the end of the first year. At the end of the second year this sum becomes

$$51(1 + 0.02) = 50(1 + 0.02)^2 = \$52.02$$

and, after x years,

$$\$50(1 + 0.02)^x$$

On the other hand, at \$5 a year, the subscription will cost \$$5x$ in x years. Hence the first form of subscription becomes more convenient than the second as soon as

$$5x \geq 50(1 + 0.02)^x$$

This is a transcendental equation for x, which, by taking logarithms on both sides, becomes

$$\log 5 + \log x \geq \log 50 + x \log 1.02$$

or

$$\log_{10} x \geq \log_{10} 10 + x \log_{10} 1.02 = 1 + 0.0086x$$

To solve this equation, plot the graphs of the logarithmic function $y_1 = \log_{10} x$ and of the straight line $y_2 = 1 + 0.0086x$, and find the abscissa of their intersection (Fig. 5·5). The two graphs meet roughly at $x = 13$; hence, as soon as $x \geq 13$ years, it pays to use the life-subscription rate.

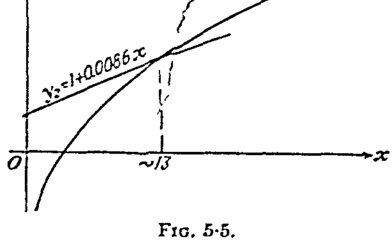


FIG. 5·5.

The logarithmic function used in this problem was taken with base 10; but, in problems involving the calculus, logarithmic functions are always taken with base $e = 2.718 \ldots$ (natural logarithms) for the

reason explained in Sec. 1·7 *b*2. Since e and 10 are positive numbers and powers of positive numbers are always positive, logarithms of negative numbers do not exist.[1] The $y = \log x$ curve lies entirely to the right of the y axis, crosses the x axis at $x = 1$, since $\log_b = 0$ whatever the base b, and approaches asymptotically the y axis as x approaches zero, since $\log_b 0 = -\infty$ whatever b.

5·6 Exponential Functions

A moneylender loans money at the interest rate of 100 per cent per year. Starting with \$1, he gets $1 + 1 = 2$ dollars at the end of the first year. If he loans again, with both capital and interest at the same rate, he gets $2 + 2 = (1 + 1)^2 = 4$ dollars at the end of the second year. At the end of x years he has $(1 + 1)^x$ dollars.

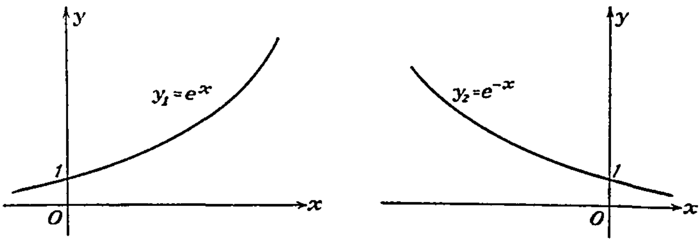


FIG. 5·6.

He then devises a better scheme: loans are given at the rate of 50 per cent for six months, and capital and interest are immediately reinvested in the same manner. The lender now gets $1 + \frac{1}{2} = 1.5$ dollars after 6 months, $(1 + \frac{1}{2})(1 + \frac{1}{2}) = (1 + \frac{1}{2})^2 = 2.25$ dollars after a year, and $(1 + \frac{1}{2})^{2x}$ dollars after x years, which is considerably more than he obtained before. By loaning at the rate of 25 per cent for 3 months, he will get, after x years, $(1 + \frac{1}{4})^{4x}$ dollars, which is still more, while by loaning at $\frac{1}{365}$ per cent per day he gets after x years the still larger sum of $(1 + \frac{1}{365})^{365x}$ dollars. Would he get indefinitely more by compounding interest for *indefinitely* shorter periods of time? This is not the case because

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e = 2.718 \cdots$$

is a finite number. The most the lender can make after x years is e^x dollars. The graph of the function $y_1 = e^x$ is given in Fig. 5·6, together with the graph of the reciprocal function $y_2 = e^{-x}$; e^x and e^{-x} are the basic *exponential functions.*

Many living quantities, like populations or cells, grow according to the following simple law: *The rate of growth is proportional, at any time,*

[1] See, however, Sec. 5·12.

to the quantity present at the time. If we call y the quantity and k the constant of proportionality between the rate and y, this law is stated mathematically by the equation

$$\frac{dy}{dt} = ky \tag{a}$$

Dividing by y and multiplying by dt both sides of Eq. (a), and integrating, we obtain

$$\int \frac{dy}{y} = \log_e y = kt + c$$

where c is a constant of integration; raising e to the power $\log_e y$,

$$e^{\log_e y} = y = e^{kt+c} = Ce^{kt}$$

where $C = e^c$ is another arbitrary constant. If, at $t = 0$, $y = y_0$, the constant C equals y_0 and

$$y = y_0 e^{kt}$$

The functions y, which grow according to the above-mentioned law, are exponential functions, and the law is sometimes called the "compound-interest law of nature."

If $y = e^x$, $x = \log_e y$. The exponential function is thus found to be the inverse logarithmic function with base e. The inverse of $x = \log_b y$ is the more general exponential function $y = b^x$. Its derivative can be computed by noticing that

$$b^x = (e^{\log_e b})^x = e^{x \log_e b}$$

and that therefore

$$\frac{db^x}{dx} = \log_e b \cdot e^{x \log_e b} = \log_e b \cdot b^x \tag{5·6·1}$$

When $b = e$,

$$\frac{de^x}{dx} = e^x \tag{5·6·2}$$

The function $y = e^x$ is the only function equal to all its successive derivatives.

5·7 Hyperbolic Functions

We wish to find the equation $y(x)$ of the curve assumed by a cable of constant weight w per unit length, hanging from two points A,B in a horizontal plane (Fig. 5·7).

Refer the cable to a system of horizontal and vertical axes x,y with origin O under the lowest point of the cable (Fig. 5·7a), and call T the tension in the cable at a section x, H the horizontal and V the vertical

components of T, and θ the angle between the tangent to the cable and the x axis.

Figure 5·7b shows the forces acting on an isolated element ds of the cable. Since ds is in equilibrium, the resultant of these forces in the horizontal direction is equal to zero and hence the horizontal component

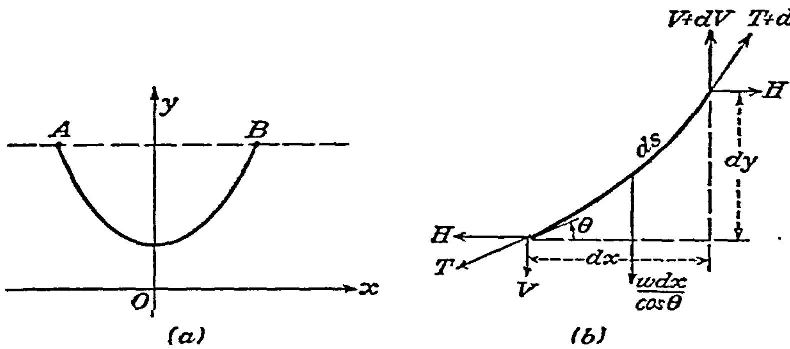


FIG. 5·7.

H of T is constant. On the other hand, the equilibrium of the vertical forces gives

$$-V + V + dV - w\,ds = 0$$

or

$$dV = w\,ds$$

But since $V = H \tan\theta$ and $ds = dx/\cos\theta$

$$dV = \frac{dV}{d\theta}\,d\theta = H\,\frac{d\tan\theta}{d\theta} = H\,\frac{d\theta}{\cos^2\theta} \qquad w\,ds = \frac{w\,dx}{\cos\theta}$$

and, substituting in the equilibrium equation,

$$\frac{d\theta}{\cos\theta} = \frac{w\,dx}{H}$$

Noticing that $\theta = 0$ at $x = 0$ and integrating between 0 and x for x and between 0 and θ for θ, we obtain

$$\int_0^\theta \frac{d\theta}{\cos\theta} = \frac{w}{H}\,x$$

The integral on the left side of this equation is tabulated, for example, in Peirce's "A Short Table of Integrals," No. 288,

$$\int_0^\theta \frac{d\theta}{\cos\theta} = \log\tan\left(\frac{1}{4}\pi + \frac{1}{2}\theta\right)\Big]_0^\theta$$

and hence

$$\log \tan \left(\frac{1}{4}\pi + \frac{1}{2}\theta\right) = \frac{wx}{H}$$

or

$$\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta\right) = e^{wx/H} \tag{a}$$

Since the cable curve is symmetrical with respect to the y axis, the value of θ at $-x$ must be equal to $-\theta$. In other words, Eq. (a) is satisfied if x is changed into $-x$ and θ into $-\theta$,

$$\tan\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\theta\right) = e^{-wx/H} \tag{b}$$

Subtracting Eq. (b) from Eq. (a) and remembering the expression for $\tan(\alpha \pm \beta)$ (Table 5·2), we find that

$$\begin{aligned} e^{wx/H} - e^{-wx/H} &= \tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta\right) - \tan\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\theta\right) \\ &= \frac{1 + \tan \frac{1}{2}\theta}{1 - \tan \frac{1}{2}\theta} - \frac{1 - \tan \frac{1}{2}\theta}{1 + \tan \frac{1}{2}\theta} = \frac{4 \tan \frac{1}{2}\theta}{1 - \tan^2 \frac{1}{2}\theta} = 2 \tan \theta \end{aligned}$$

or

$$\tan \theta = \tfrac{1}{2}\left(e^{wx/H} - e^{-wx/H}\right) \tag{c}$$

Remembering that $\tan \theta = dy/dx$ and integrating between 0 and x both sides of Eq. (c), we obtain

$$\begin{aligned} \int_0^x \frac{dy}{dx}\,dx = y(x) - y(0) &= \frac{1}{2}\frac{H}{w}\left[e^{wx/H} + e^{-wx/H}\right]_0^x \\ &= \frac{1}{2}\frac{H}{w}\left(e^{wx/H} + e^{-wx/H}\right) - \frac{H}{w} \end{aligned}$$

If we finally take the origin O at a distance H/w from the lowest point of the cable, *i.e.*, make $y(0) = H/w$, we obtain the equation of the cable in the form

$$y = \frac{H}{w}\,\frac{e^{wx/H} + e^{-wx/H}}{2} \tag{d}$$

The curve assumed by the cable is called a *catenary* and Eqs. (c) and (d) show that the catenary and its slope are particular combinations of exponential functions. These two combinations occur so often in engineering problems that they have been tabulated and given particular names; the function

$$y = \tfrac{1}{2}(e^x - e^{-x}) = \sinh x \tag{5·7·1}$$

is called the *hyperbolic sine of x;* the function

$$y = \tfrac{1}{2}(e^x + e^{-x}) = \cosh x \tag{5·7·2}$$

is called the *hyperbolic cosine of* x. The function

$$y = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh x \tag{5·7·3}$$

ratio of sinh x to cosh x, is called the *hyperbolic tangent of* x. The graphs of these three functions appear in Fig. 5·8.

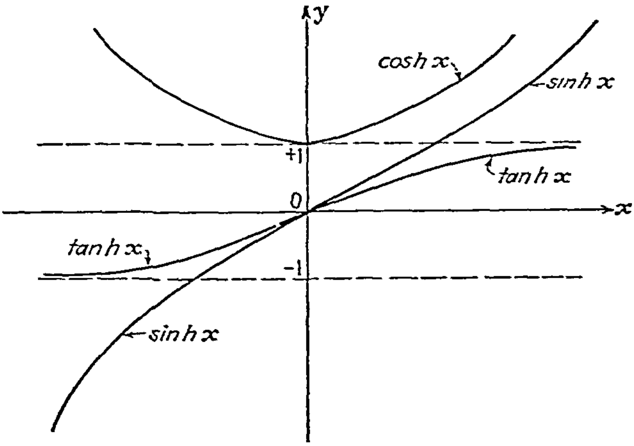


FIG. 5·8.

The hyperbolic functions present some remarkable analogies with the corresponding circular functions:

1. From their definitions,

$$\sinh 0 = \frac{e^0 - e^{-0}}{2} = 0 \qquad \cosh 0 = \frac{e^0 + e^{-0}}{2} = 1$$

just as $\sin 0 = 0$ and $\cos 0 = 1$.

2. Differentiating Eqs. (5·7·1) and (5·7·2),

$$\frac{d}{dx}(\sinh x) = \frac{e^x + e^{-x}}{2} = \cosh x \tag{5·7·4}$$

$$\frac{d}{dx}(\cosh x) = \frac{e^x - e^{-x}}{2} = \sinh x \tag{5·7·5}$$

These relations are analogous to those between $\sin x$ and $\cos x$, apart from a sign factor.

3. If we set

$$x = \cos \alpha \qquad y = \sin \alpha \tag{e}$$

and determine the locus of points defined by Eq. (*e*), by elimination of α between these two equations, we find that this locus is the circle

$$x^2 + y^2 = 1$$

(For this reason the trigonometric functions are also called circular.) Similarly, if we set

$$x = \cosh \alpha \qquad y = \sinh \alpha \tag{f}$$

and eliminate α by squaring and subtracting, we find

$$\begin{aligned} x^2 - y^2 &= \cosh^2 \alpha - \sinh^2 \alpha = \tfrac{1}{4}(e^{\alpha} + e^{-\alpha})^2 - \tfrac{1}{4}(e^{\alpha} - e^{-\alpha})^2 \\ &= \tfrac{1}{4}(e^{2\alpha} + 2e^{\alpha}e^{-\alpha} + e^{-2\alpha}) - \tfrac{1}{4}(e^{2\alpha} - 2e^{\alpha}e^{-\alpha} + e^{-2\alpha}) = 1 \end{aligned}$$

i.e., the locus of points defined by Eq. (f) is a rectangular hyperbola. For this reason, $\sinh x$ and $\cosh x$ are called hyperbolic functions.

The hyperbolic functions are single-valued, *nonperiodic* (for x real), and continuous. $\sinh x$ and $\tanh x$ are odd functions; $\cosh x$ is even. Identities similar to those existing for the trigonometric functions hold for the hyperbolic functions. The most important are

$$\cosh^2 x - \sinh^2 x = 1 \tag{5·7·6}$$
$$\sinh (x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \tag{5·7·7}$$
$$\cosh (x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y \tag{5·7·8}$$

The inverse hyperbolic functions are indicated by the symbols

$$\begin{aligned} \operatorname{argsinh} x &= \sinh^{-1} x \\ \operatorname{argcosh} x &= \cosh^{-1} x \\ \operatorname{argtanh} x &= \tanh^{-1} x \end{aligned}$$

argsinh x and argtanh x are single-valued; argcosh x is double-valued.

5·8 Binomial Expansion

a. A cable of length L ft is attached to two points in a horizontal plane a ft apart and loaded uniformly with a load of w lb *per horizontal foot* (Fig. 5·9). By assumption, the weight of the cable is negligible in comparison with the load, and the sag f is small in comparison with a. We wish to find the equation of the curve assumed by the cable.

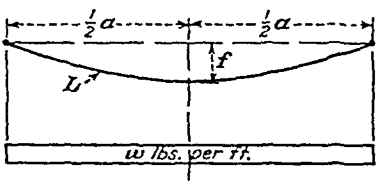


FIG. 5·9.

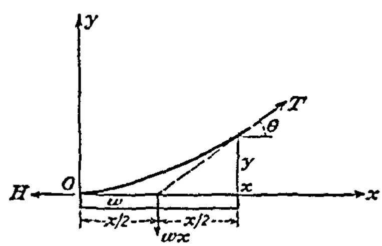


FIG. 5·10.

Refer the cable to a set of axes with origin at its lowest point [the x axis horizontal and positive to the right, the y axis vertical and positive upward (Fig. 5·10)], and call H the tension in the cable at its lowest

point, T the tension and θ the angle, between the tangent to the cable and the x axis, at a section x.

Considering the equilibrium of the horizontal and vertical forces applied to a portion of the cable between O and x, we find that

$$-H + T \cos \theta = 0$$
$$-wx + T \sin \theta = 0$$

and, taking the ratio of these two equations, that

$$\tan \theta = \frac{wx}{H} \tag{a}$$

We now notice that the portion of cable between O and x is in equilibrium under the action of three forces, H, T, and wx. Hence these three forces must be concurrent; and since wx crosses H at $x/2$, T must also cross the x axis at this point. It follows from Fig. 5·10 that

$$\tan \theta = \frac{y}{x/2}$$

and by Eq. (a) that

$$y = \frac{w}{2H} x^2$$

In order to eliminate the unknown force H, notice that, when $x = a/2$, $y = f$,

$$f = \frac{wa^2}{8H}$$

from which

$$\frac{w}{H} = \frac{8f}{a^2}$$

and

$$y = \frac{4fx^2}{a^2} \tag{b}$$

When f and a are known, Eq. (b) gives the curve assumed by the cable, which is a quadratic parabola.

In the present problem, f is unknown, while the length L and the span a of the cable are given and f must be expressed in terms of L and a before Eq. (b) can be used. To find this relationship, we compute the length of the curve (b) between $x = -a/2$ and $x = a/2$.

An infinitesimal element ds of the curve has a length

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dx^2[1 + (y')^2]} \doteq dx \sqrt{1 + \left(\frac{8fx}{a^2}\right)^2}$$

and the total length L is given by the integral

$$L = \int_{-a/2}^{a/2} \sqrt{1 + \left(\frac{8fx}{a^2}\right)^2}\, dx = 2\int_0^{a/2} \sqrt{1 + \left(\frac{8fx}{a^2}\right)^2}\, dx \qquad (c)$$

since the cable curve is symmetrical about the y axis. An easy way of evaluating this integral is to expand the integrand by the binomial theorem.

b. It is well known from algebra that, when n is a positive integer, the nth power of the binomial $a \pm b$ can be computed by means of the following formula (*Newton's binomial theorem*):

$$(a \pm b)^n = a^n \pm \frac{n}{1} a^{n-1}b + \frac{n(n-1)}{1 \cdot 2} a^{n-2}b^2 \pm \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}b^3 + \cdots \qquad (5\cdot8\cdot1)$$

The various terms of Eq. (5·8·1) are certain products of constants times decreasing powers of a and increasing powers of b, such that the sum of the exponents of a and b is always n. *When n is a positive integer*, the expansion contains $(n + 1)$ terms and the constants are given by *Pascal's triangle* (known at least 700 years before Pascal rediscovered it),

n						
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1
. .	. .	. .	. .	. .	. .	. .

in which every row begins and ends with unity and the successive coefficients are the sum of the two adjacent coefficients of the preceding row. The general expression for the binomial coefficient $C_{m,n}$ of the term $a^{n-m}b^m$ in the expansion of $(a + b)^n$ is

$$C_{m,n} = \frac{n!}{(n-m)!m!} \qquad (n,m \text{ integers}) \qquad (5\cdot8\cdot2)$$

where the symbol $n!$ (n factorial) stands for

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$$

and, *by definition*, $0! = 1$.

Equation (5·8·1) can be proved to hold, not only for positive integral exponents, but for any real exponent n, provided that $b < a$. When n is

not a positive integer, Eq. (5·8·1) contains an *infinite number of terms* and becomes what is called an *infinite series.*

c. To expand by the binomial theorem the integrand of Eq. (*c*), for simplicity of notation let

$$\frac{8f}{a^2} = k; \qquad (kx)^2 = y$$

and apply Eq. (5·8·1) with $a = 1$, $b = y$, and $n = \frac{1}{2}$.

$$\begin{aligned}\sqrt{1 + \left(\frac{8fx}{a^2}\right)^2} &= (1 + y)^{1/2} \\ &= 1^{1/2} + \frac{\frac{1}{2}}{1} 1^{-1/2} y + \frac{\frac{1}{2}(-\frac{1}{2})}{1 \cdot 2} 1^{-3/2} y^2 \\ &\qquad + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1 \cdot 2 \cdot 3} 1^{-5/2} y^3 + \cdots \\ &= 1 + \tfrac{1}{2}(kx)^2 - \tfrac{1}{8}(kx)^4 + \tfrac{1}{16}(kx)^6 - \cdots\end{aligned}$$

Integrating this series term by term,

$$\begin{aligned}\int_0^{a/2} \sqrt{1 + (kx)^2}\, dx &= \left[x + \frac{1}{2} k^2 \frac{x^3}{3} - \frac{1}{8} k^4 \frac{x^5}{5} + \frac{1}{16} k^6 \frac{x^7}{7} - \cdots\right]_0^{a/2} \\ &= \frac{1}{2} a + \frac{\frac{1}{2} k^2 a^3}{8 \cdot 3} - \frac{1}{8} \frac{k^4 a^5}{32 \cdot 5} + \frac{1}{16} \frac{k^6 a^7}{128 \cdot 7} - \cdots \\ &= \frac{a}{2}\left[1 + \frac{8}{3}\left(\frac{f}{a}\right)^2 - \frac{32}{5}\left(\frac{f}{a}\right)^4 + \frac{128}{7}\left(\frac{f}{a}\right)^6 - \cdots\right]\end{aligned}$$

and by Eq. (*c*) the length L of the cable curve becomes

$$L = a\left[1 + \frac{8}{3}\left(\frac{f}{a}\right)^2 - \frac{32}{5}\left(\frac{f}{a}\right)^4 + \frac{128}{7}\left(\frac{f}{a}\right)^6 - \cdots\right]$$

When the ratio f/a is small in comparison with unity, its higher powers can be neglected and the length L can be approximated by

$$L = a\left[1 + \frac{8}{3}\left(\frac{f}{a}\right)^2 - \frac{32}{5}\left(\frac{f}{a}\right)^4\right]$$

as was done in Sec. 3·3. Solving this equation for the sag in terms of a and L, the cable equation (*b*) can be written in each case. For instance, with $a = 99$ ft and $L = 100$ ft, $f = 6.06$ ft (see Sec. 3·3) and

$$y = 0.00247x^2$$

d. Integration is one of the many fields in which the binomial expansion can be usefully applied; approximate numerical computations

is another. For instance, applying Eq. (5·8·1) to the function

$$y = \frac{1}{(1 + x)},$$

where x is less than 1, we find

$$\frac{1}{1 + x} = (1 + x)^{-1} = 1 - x + x^2 - x^3 + \cdots \tag{5·8·3}$$

and, for small values of x,

$$\frac{1}{1 + x} \doteq 1 - x$$

Thus $1/1.02 \doteq 0.98$ with an error of 0.04 per cent.

Changing x into $-x$ in Eq. (5·8·3),

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots \tag{5·8·4}$$

and, for small values of x,

$$\frac{1}{1 - x} \doteq 1 + x$$

Thus $1/0.99 \doteq 1.01$ with an error of 0.01 per cent.

Similarly, to compute the square root of a number N, of which m is an approximate root, let

$$N = m^2 + x \qquad (x < m^2)$$

and apply Eq. (5·8·1) with $a = m^2$, $b = x$, and $n = \frac{1}{2}$.

$$\begin{aligned} \sqrt{N} &= \sqrt{m^2 + x} = (m^2 + x)^{\frac{1}{2}} \\ &= (m^2)^{\frac{1}{2}} + \tfrac{1}{2}(m^2)^{-\frac{1}{2}}x - \tfrac{1}{8}(m^2)^{-\frac{3}{2}}x^2 + \cdots \\ &= m + \frac{x}{2m} - \frac{x^2}{(2m)^3} + \cdots \end{aligned}$$

For instance,

$$\begin{aligned} \sqrt{5} = \sqrt{4 + 1} = \sqrt{2^2 + 1} &= 2 + \frac{1}{2 \times 2} - \frac{1}{(2 \times 2)^3} + \cdots \\ &\doteq 2 + 0.250 - 0.0156 = 2.234 \end{aligned}$$

with an error of less than 0.1 per cent ($\sqrt{5} = 2.236$).

Similarly, to compute the cube root of a number of which m is an approximate cube root, let

$$N = m^3 + x \qquad (x < m^3)$$

and apply Eq. (5·8·1) with $a = m^3$, $b = x$, and $n = \frac{1}{3}$.

$$\begin{aligned}\sqrt[3]{N} &= \sqrt[3]{m^3 + x} = (m^3 + x)^{1/3} \\ &= (m^3)^{1/3} + \frac{1}{3}(m^3)^{-2/3}x + \frac{(\frac{1}{3})(-\frac{2}{3})}{1 \cdot 2}(m^3)^{-5/3}x^2 + \cdots \\ &= m + \frac{x}{3m^2} - \frac{x^2}{9m^5} + \cdots\end{aligned}$$

For instance,

$$\begin{aligned}\sqrt[3]{7} &= \sqrt[3]{8 - 1} = \sqrt[3]{2^3 - 1} \\ &= 2 + \frac{-1}{3 \cdot 2^2} - \frac{(-1)^2}{9 \cdot 2^5} + \cdots \\ &\doteq 2 - 0.0833 - 0.00347 = 1.913\end{aligned}$$

which is correct to the last figure.

5·9 Maclaurin's and Taylor's Series

a. The fact that simple functions, like $1/(1 + x)$, $\sqrt{1 + x^2}$, etc., may be expanded into power series by means of the binomial theorem leads one to investigate whether other functions, like $\sin x$ or e^x, could also be so expanded. To give an answer to this question we shall consider a specific function, say $\sin x$, and assume that: (1) this function can be expanded into an infinite series of increasing powers of x with constant coefficients, *i.e.*, that coefficients $a_0, a_1, a_2, \ldots, a_n, \ldots$ may be found such that the equation

$$f(x) = \sin x = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots \tag{a}$$

be true for all values of x, at least within a given interval; (2) the series obtained from Eq. (*a*) by successive differentiation, term by term, is equal to the successive derivatives of $\sin x$.

Since the left-hand member of Eq. (*a*) is equal to its right-hand member for all values of x, it must be equal in particular for $x = 0$ and hence

$$f(0) = \sin 0 = 0 = a_0$$

from which

$$a_0 = f(0) = 0$$

Similarly, differentiating both sides of Eq. (*a*),

$$f'(x) = \cos x = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots \tag{b}$$

and, setting $x = 0$,

$$f'(0) = \cos 0 = 1 = a_1$$

from which

$$a_1 = f'(0) = 1 = \frac{1}{1!}$$

Differentiating Eq. (*b*),

$$f''(x) = -\sin x = 2a_2 + 2 \cdot 3a_3x + 3 \cdot 4a_4x^2 + 4 \cdot 5a_5x^3 + \cdots \quad (c)$$

and, setting $x = 0$,

$$f''(0) = -\sin 0 = 0 = 2a_2$$

from which

$$a_2 = \frac{f''(0)}{2} = \frac{f''(0)}{2!} = 0$$

Differentiating Eq. (*c*),

$$f'''(x) = -\cos x = 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4x + 3 \cdot 4 \cdot 5a_5x^2 + \cdots \quad (d)$$

and, setting $x = 0$,

$$f'''(0) = -\cos 0 = -1 = 2 \cdot 3a_3$$

from which

$$a_3 = \frac{f'''(0)}{2 \cdot 3} = \frac{f'''(0)}{3!} = -\frac{1}{3!}$$

This procedure can be applied indefinitely and leads to the so-called *Maclaurin's expansion* of $f(x)$,

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \quad (5{\cdot}9{\cdot}1)$$

When $f(x) = \sin x$, Eq. (5·9·1) gives

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots \quad (5{\cdot}9{\cdot}2)$$

Since $\sin x$ is an odd function, its expansion contains only odd powers of x.

Applying Eq. (5·9·1) to $f(x) = \cos x$, we obtain, similarly,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \quad (5{\cdot}9{\cdot}3)$$

Since $\cos x$ is even, its expansion contains only even powers of x.

In expanding $y = e^x$, we notice that the successive derivatives of y are all equal to e^x and that $e^0 = 1$; hence, by Eq. (5·9·1),

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \quad (5{\cdot}9{\cdot}4)$$

and, changing x into $-x$ on both sides of Eq. (5·9·4),

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \cdots \quad (5{\cdot}9{\cdot}5)$$

All the preceding expansions are well suited for the evaluation of the corresponding functions in the neighbourhood of zero, since for small values of x the successive terms of the series decrease rapidly. For example, by means of Eq. (5·9·4),

$$\begin{aligned} e^{0.1} &= 1 + 0.1 + \frac{0.01}{2} + \frac{0.001}{6} + \frac{0.0001}{24} + \cdots \\ &= 1 + 0.1 + 0.005 + 0.000167 + 0.000004 + \cdots = 1.105171 \end{aligned}$$

which is correct to six decimal places.

If, on the other hand, the value of the function is required for large values of x, the number of terms to be taken into account to obtain a given degree of accuracy may easily become so large as to make the computation prohibitive. In this case, instead of expanding the function in terms of increasing powers of x, the function may be expanded in terms of increasing powers of $(x - a)$, where a is a suitable number. For this purpose we set

$$\begin{aligned} f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 \\ + c_4(x - a)^4 + \cdots + c_n(x - a)^n + \cdots \qquad (e) \end{aligned}$$

where the $c_i (i = 0, 1, 2, 3, \ldots)$ are unknown constants. Letting $x = a$ in Eq. (e), we find immediately

$$f(a) = c_0$$

while, differentiating successively both sides of Eq. (e) with respect to x and letting $x = a$ in each derivative, we obtain

$$\begin{aligned} f'(a) &= c_1 \qquad &\text{or} \qquad c_1 &= \frac{f'(a)}{1!} \\ f''(a) &= 2c_2 \qquad &\text{or} \qquad c_2 &= \frac{f''(a)}{2!} \\ &\cdots\cdots &\cdots & \\ f^{(n)}(a) &= n!c_n \qquad &\text{or} \qquad c_n &= \frac{f^{(n)}(a)}{n!} \end{aligned}$$

by means of which the expansion of $f(x)$ becomes

$$\begin{aligned} f(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 \\ + \frac{f'''(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \cdots \qquad (5·9·6) \end{aligned}$$

This expansion is called the power-series expansion of $f(x)$ about $x = a$ or *Taylor's expansion of* $f(x)$ *about* $x = a$. It will be noticed that Maclaurin's expansion is a particular case of Taylor's expansion with $a = 0$.

For example, applying Eq. (5·9·6) to $y = \log_e x$ with $a = 1$, we obtain

$$\begin{aligned} f(x) &= \log_e x & f(1) &= \log_e 1 = 0 \\ f'(x) &= \frac{1}{x} & f'(1) &= \frac{1}{1} = 1 \\ f''(x) &= \frac{-1}{x^2} & f''(1) &= -1 \\ f'''(x) &= \frac{2}{x^3} & f'''(1) &= 2 \\ f^{iv}(x) &= \frac{-2\cdot 3}{x^4} & f^{iv}(1) &= -6 \end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\log_e x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots \qquad (5\cdot 9\cdot 7)$$

It may be well to notice that $\log x$ cannot be expanded into a Maclaurin's series because the first term of the expansion, $f(0)$, would be infinite. By means of the first four terms of Eq. (5·9·7) we may compute

$$\begin{aligned} \log_e 1.1 &= 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} \\ &= 0.1 - 0.005 + 0.000333 - 0.000025 = 0.095308 \end{aligned}$$

which is correct within one unit in the sixth decimal digit.

b. In the numerical evaluation of a function by means of Taylor's or Maclaurin's series, only a finite number of terms can be taken into account, and an error is thus introduced into the computations.

When the first n terms of a Taylor's series are taken into account, *i.e.*, when the last term considered is

$$\frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}$$

it can be proved[1] that the sum of *all* the remaining terms of the series is equal to

$$R_n = \frac{f^{(n)}(\bar{x})}{n!}(x-a)^n \qquad (5\cdot 9\cdot 8)$$

where $\bar{x}$ is a particular value of x between a and x. R_n is called the *remainder of the series after the nth term* and has the same form as the $(n+1)$th term of the series, but here the nth derivative must be taken *at $\bar{x}$ rather than at a.*

[1] See for example H. M. Bacon, "Differential and Integral Calculus," p. 304. McGraw-Hill Book Company, Inc., New York, 1942.

For instance, the remainder of the Maclaurin's series of $y = e^x$ is

$$R_n = \frac{e^{\bar{x}}}{n!} x^n$$

since $f^{(n)}(x) = e^x$ and $a = 0$ in a Maclaurin's expansion. If it were possible to determine by elementary methods the value $\bar{x}$ of x appearing in the remainder formula, the precise value of R_n could be found and the sum of the series obtained. Unfortunately, in most cases $\bar{x}$ cannot be determined by elementary means, and the remainder formula can be used only to find an *upper bound*, or limit, to the error made in considering the first n terms of the expansion.

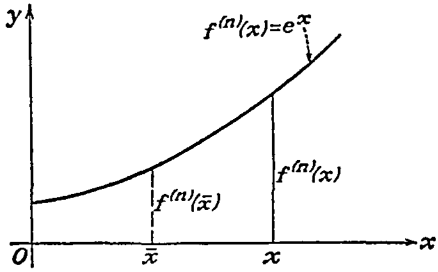


FIG. 5·11.

This is done by finding the largest value that $f^{(n)}(x)$ can have in the interval (a,x) and by substituting this largest value for $f^{(n)}(\bar{x})$ in the remainder formula. For instance, in the Maclaurin's expansion of e^x, $f^{(n)}(x) = e^x$ is an *increasing* function of x; and since $0 \le \bar{x} \le x$, $f^{(n)}(x)$ is certainly no smaller than $f^{(n)}(\bar{x})$ (Fig. 5·11). The remainder is therefore not larger than

$$R_n \le \frac{f^{(n)}(x)}{n!} x^n = \frac{e^x x^n}{n!}$$

Thus the error made in stopping the Maclaurin's expansion of e^x at $x = 0.1$ after five terms is less than or equal to

$$R_n \le \left.\frac{e^x x^5}{5!}\right]_{x=0.1} = \frac{1.105171}{120} 0.00001 = 0.92 \times 10^{-7}$$

The approximate value of e^x, obtained by considering the first five terms of the expansion, can be used in computing this error since the remainder gives only the *order of magnitude* of the error.

Similarly, the error made in stopping the Maclaurin's expansion of $\sin x$ after the term $x^3/3!$ (the fourth term of the expansion) is given by

$$R_4 = \frac{f^{(4)}(\bar{x})}{4!} x^4 = \frac{\sin \bar{x}}{4!} x^4$$

Since $\sin x$ is an oscillatory function, it may be cumbersome to find at what point, between 0 and x, $\sin x$ becomes maximum; but since $\sin x$ is always less than 1 in absolute value, $\sin \bar{x}$ is no larger than 1 and a simple

bound for the remainder becomes

$$R_n \leq \frac{x^4}{4!}$$

If we notice that the fifth term $(x^4/4!)$ does not appear in the expansion of $\sin x$, we may compute the remainder after the fifth term,

$$R_5 = \frac{f^{(5)}(\bar{x})}{5!} x^5 = \frac{\cos \bar{x}}{5!} x^5 \leq \frac{x^5}{5!}$$

and use this smaller bound for the error after the term $x^3/3!$

To compute the remainder after the fifth term of Taylor's expansion of $\log_e x$ [Eq. (5·9·7)], substitute

$$f^{(5)}(\bar{x}) = \frac{d^5}{dx^5} \log_e x \Big]_{x=\bar{x}} = \frac{2 \cdot 3 \cdot 4}{\bar{x}^5}$$

in Eq. (5·9·8).

$$R_5 = \frac{2 \cdot 3 \cdot 4}{\bar{x}^5} \frac{(x-1)^5}{5!} = \frac{(x-1)^5}{5\bar{x}^5}$$

The largest value of $f^{(5)}(\bar{x})$ in the interval $(1, x)$ is found at its left end $\bar{x} = 1$, since $f^{(5)}(x)$ is a decreasing function of x; hence an upper bound for the remainder is given by

$$R_5 \leq \frac{(x-1)^5}{5}$$

At $x = 1.1$ this gives

$$R_5 \Big]_{x=1.1} \leq \frac{(1.1-1)^5}{5} = 2 \times 10^{-6}$$

An upper bound for the error after the nth term in the power-series expansion of other elementary functions can usually be found without difficulty by these procedures.

5·10 Convergence of Series

a. In the preceding sections, use has been made of infinite series without inquiring about the meaning of the expression "sum of an infinite number of terms." If we bear in mind that, even when a very large number of terms of an infinite series is taken into account, an *infinite* number of terms is dropped, the necessity of defining the sum of a series will appear obvious.

The sum of the first n terms of a numerical series,

$$a_0 + a_1 + a_2 + \cdots + a_n + \cdots \tag{5·10·1}$$

is called its partial sum S_n.

$$S_n = a_0 + a_1 + a_2 + \cdots + a_{n-1} \tag{5·10·2}$$

By definition, the sum S of an infinite series is the limit of S_n as n approaches infinity.

$$S = \lim_{n \to \infty} S_n \tag{5·10·3}$$

When the number S exists and is finite, the series is said to be *convergent;* when S is infinity, the series is said to be *divergent;* when S does not exist, the series is called *oscillating* or divergent.

For instance, the convergence of the *geometric series of ratio* r,

$$1 + r + r^2 + \cdots + r^{n-1} + r^n + \cdots \tag{5·10·4}$$

can be investigated by writing

$$S_n = 1 + r + r^2 + \cdots + r^{n-1}$$

multiplying S_n by r,

$$rS_n = r + r^2 + r^3 + \cdots + r^{n-1} + r^n$$

and subtracting this last equation from S_n,

$$S_n(1 - r) = 1 - r^n$$

Thus

$$S_n = \frac{1 - r^n}{1 - r}$$

and, taking the limit as n approaches infinity,

$$\lim_{n \to \infty} S_n = \begin{cases} 1/(1 - r) & \text{when } |r| < 1 \\ \pm\infty & \text{when } |r| > 1 \end{cases}$$

When $r = 1$, the series is divergent, since it is the sum of an infinite number of 1's; when $r = -1$, the series is oscillating, since the partial sums are all equal to 1 for n odd and to zero for n even. Hence the geometric series is convergent only for $|r| < 1$.

It may erroneously be thought that, if the terms of a series become indefinitely smaller, the series will necessarily converge. The simple example of the *harmonic series*

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots \tag{5·10·5}$$

whose terms become indefinitely smaller, will show that this condition is not sufficient to guarantee the convergence of the series.

If in the harmonic series we substitute

$$\tfrac14 + \tfrac14 = 2 \times \tfrac14 = \tfrac12 \quad \text{for} \quad \tfrac13 + \tfrac14$$
$$\tfrac18 + \tfrac18 + \tfrac18 + \tfrac18 = 4 \times \tfrac18 = \tfrac12 \quad \text{for} \quad \tfrac15 + \tfrac16 + \tfrac17 + \tfrac18$$
$$8 \times \tfrac{1}{16} = \tfrac12 \quad \text{for} \quad \tfrac19 + \tfrac{1}{10} + \tfrac{1}{11} + \tfrac{1}{12} + \tfrac{1}{13} + \tfrac{1}{14} + \tfrac{1}{15} + \tfrac{1}{16}$$

. . . .

we obtain a new series, which is certainly smaller than the harmonic series,

$$1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} \cdots$$

But this series is divergent, because it is the sum of an infinite number of equal finite terms; hence the harmonic series, which is larger, is certainly divergent. The first billion terms of the harmonic series add up to about 21, but *all* the terms add up to infinity.

In what follows, infinite series like that of Eq. (5·10·1) will be symbolized by the Greek letter Σ (capital sigma).

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots + a_n + \cdots$$

b. The definition oɪ sum given by Eqs. (5·10·2) and (5·10·3) gives a method for checking the convergence of series, but this method is seldom easily applicable. Hence other and simpler *tests* have been devised to investigate the convergence of a series, of which a few will be stated without proof.

1. *The Comparison Test.* "A series $\sum_{n=0}^{\infty} a_n$ is convergent if its terms a_n are, at least from a given term on, respectively smaller than or equal to the terms b_n of a converging series $\sum_{n=0}^{\infty} b_n$."

For instance, the series

$$1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^n} \qquad (a)$$

has, from the second term on, terms respectively smaller than the terms of the geometric series of ratio $\frac{1}{2}$,

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots = \sum_{n=0}^{\infty} \frac{1}{2^n}$$

This geometric series is convergent because $r < 1$ (see Sec. 5·10 *a*); hence Eq. (*a*) is convergent.

"A series $\sum_{n=0}^{\infty} a_n$ is divergent if its terms a_n are, at least from a given term on, respectively larger than or equal to the terms c_n of a diverging series $\sum_{n=0}^{\infty} c_n$."

For instance, the series

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

is divergent because its terms are larger than the corresponding terms of the harmonic series, which is known to be divergent.

The geometric series of ratio $r < 1$ and the k series

$$1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^k} \tag{5·10·6}$$

which converges for $k > 1$ and diverges for $k \le 1$, are often used in the comparison test. The test does not apply when the terms of the series are larger than the terms of a converging series or smaller than the terms of a diverging series.

2. *The Ratio Test.* "A series $\sum_{n=0}^{\infty} a_n$ is convergent or divergent depending upon whether the limit

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \tag{5·10·7}$$

is less than or more than 1. When $L = 1$, the test fails."

For example, the series

$$1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \frac{5}{2^4} + \cdots = \sum_{n=1}^{\infty} \frac{n}{2^{n-1}}$$

is convergent, since

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)/2^n}{n/2^{n-1}}\right| = \lim_{n\to\infty} \left|\frac{n+1}{2^n}\,\frac{2^{n-1}}{n}\right|$$
$$= \lim_{n\to\infty} \left|\frac{1}{2}\,\frac{n+1}{n}\right| = \lim_{n\to\infty} \left|\frac{1}{2}\,\frac{1+(1/n)}{1}\right| = \frac{1}{2} < 1$$

The series

$$\frac{1}{k} + \frac{2!}{k^2} + \frac{3!}{k^3} + \cdots = \sum_{n=1}^{\infty} \frac{n!}{k^n}$$

is divergent, since

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{k^{n+1}}\,\frac{k^n}{n!}\right| = \lim_{n\to\infty} \left|\frac{n+1}{k}\right| = \infty > 1$$

This test fails when applied to the series

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

since

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{n(n+1)}{(n+1)(n+2)}\right|=\lim_{n\to\infty}\left|\frac{n}{n+2}\right|$$
$$=\lim_{n\to\infty}\left|\frac{1}{1+(2/n)}\right|=1$$

but this series is convergent since, by the comparison test, its terms are smaller than the terms of the k series (5·10·6) with $k = 2$.

3. *Alternating-series Test.* "An alternating series

$$a_0-a_1+a_2-a_3+\cdots=\sum_{n=0}^{\infty}(-1)^n a_n$$

where all the a_n are positive, is convergent if at least from a certain n on

$$a_{n+1}<a_n \qquad \text{and} \qquad \lim_{n\to\infty} a_n = 0."$$

For example, the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} \tag{b}$$

is convergent, because

$$\frac{1}{n+1}<\frac{1}{n} \qquad \text{and} \qquad \lim_{n\to\infty}\frac{1}{n}=0$$

The error made by stopping the summation of an alternating series after n terms can be proved to be smaller than the $(n + 1)$th term.

4. *Power-series Test.* The ratio test can be used to determine the *range of convergence* of a power series, *i.e.*, the range of values of x for which the power series will converge.

A power series $\sum_{n=0}^{\infty} a_n x^n$, in which x is given a particular value, becomes a series of numbers, to which the ratio test can be applied. The limit (5·10·7) becomes, in this case,

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right|=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}x\right|=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||x|$$

and, letting,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r \tag{5·10·8}$$

the ratio-test condition for convergence becomes

$$L = r|x| < 1 \qquad \text{or} \qquad |x| < \frac{1}{r}$$

Hence the power series will be convergent for those values of x which satisfy the condition

$$-\frac{1}{r} < x < \frac{1}{r} \tag{5·10·9}$$

For instance, the series

$$1 + \frac{x}{3\cdot 1} + \frac{x^2}{3^2\cdot 2} + \frac{x^3}{3^3\cdot 3} + \frac{x^4}{3^4\cdot 4} + \cdots = 1 + \sum_{n=1}^{\infty}\frac{x^n}{3^n n}$$

has a limit r equal to

$$\begin{aligned} r &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^n n}{3^{n+1}(n+1)}\right| = \lim_{n\to\infty}\frac{1}{3}\left|\frac{n}{n+1}\right| \\ &= \lim_{n\to\infty}\frac{1}{3}\left|\frac{1}{1+(1/n)}\right| = \frac{1}{3} \end{aligned}$$

and is therefore convergent for

$$-3 < x < 3$$

The series expansion of e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

has a ratio r equal to

$$r = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{1}{n+1}\right| = 0$$

Hence it is convergent for

$$\frac{-1}{0} = -\infty < x < \infty = \frac{1}{0}$$

i.e., for all values of x.

The series expansions of e^{-x}, sin x, cos x, sinh x, and cosh x are also convergent for all values of x.

5·11 Summation of Series

Whenever a series is convergent, it would be convenient to write its sum in finite terms, particularly when the convergence of the series is slow. Unfortunately this is not always possible, for a power series often represents a *new* type of nonelementary function. Thus, the series

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4(2!)^2} - \cdots + \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2} + \cdots$$

which can be proved to be convergent for $|x| < \infty$, represents a nonelementary function called the *Bessel function of order zero of the first kind,* the properties of which are well known but which cannot be represented by means of elementary functions in finite terms.[1]

The reader is finally reminded that, while infinite series, in general, cannot be dealt with as if they were sums of a finite number of terms, converging power series can be added, subtracted, multiplied, divided, differentiated, and integrated term by term like polynomials.

5·12 Elementary Functions of a Complex Variable

The so-called "elementary" functions (algebraic, trigonometric, logarithmic, exponential, and hyperbolic) have been considered thus far as functions of a variable x that could take only *real* values. But just as the concept of number has been extended to include imaginary and complex numbers, it is found convenient in some fields of applied mathematics (for example, electrical engineering, elasticity, and aerodynamics) to extend the concept of function to include functions of a complex variable $z = x + iy$.

The value of a *complex* polynomial, like

$$P(z) = 3z^2 + 2z + 1$$

is found by substituting for z its value $x + iy$ and performing the operations indicated.

$$\begin{aligned} P(x + iy) &= 3(x + iy)^2 + 2(x + iy) + 1 \\ &= (3x^2 + 2x - 3y^2 + 1) + i(6xy + 2y) \end{aligned}$$

$P(z)$ is thus found to be a complex number with real part $3x^2 + 2x - 3y^2 + 1$ and imaginary part $6xy + 2y$.

But the same simple interpretation is not available if we wish to find the meaning of the complex exponential $e^z = e^{x+yi}$. In this case, consider first, for simplicity's sake, the imaginary exponential e^{iz}, which

[1] Most of the known series have been tabulated in "Summation of Series" by L. B. W. Jolley (Chapman & Hall, Ltd., London, 1925), which engineers will find it useful to consult.

as yet has no meaning, and *define* its value as the value of the right-hand member of the series expansion of e^x [Eq. (5·9·4)], in which x is replaced by ix.

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots \qquad (a)$$

Since by definition

$$i^2 = -1, \qquad i^3 = i^2 i = -i, \qquad i^4 = i^2 i^2 = 1, \qquad i^5 = i^4 i = i, \qquad \ldots$$

assembling in two separate brackets all the real and all the imaginary terms of Eq. (a), we obtain

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

which, upon remembering the power-series expansions of $\cos x$ and $\sin x$ [Eqs. (5·9·2) and (5·9·3)], reduces to

$$e^{ix} = \cos x + i \sin x \qquad (5\cdot12\cdot1)$$

(*Euler's formula*). e^{ix} is thus found to be a complex number with real part $\cos x$ and imaginary part $\sin x$.

Changing x into $-x$, Eq. (5·12·1) becomes

$$e^{-ix} = \cos x - i \sin x \qquad (5\cdot12\cdot2)$$

since $\cos x$ is an even and $\sin x$ an odd function of x.

The complex exponential function e^{x+yi} can now be *defined* by applying *formally* the rule for the summation of exponents.

$$e^z = e^{x+yi} = e^x e^{yi} = e^x(\cos y + i \sin y) \qquad (5\cdot12\cdot3)$$

e^z is thus found to be a complex number of real part $e^x \cos y$ and imaginary part $e^x \sin y$.

The other elementary functions can also be defined in the complex field by means of their expansions, but their definition can be obtained easily by means of Euler's formulas. Adding Eqs. (5·12·1) and (5·12·2) and dividing by 2, we find that

$$\tfrac{1}{2}(e^{ix} + e^{-ix}) = \cos x \qquad (b)$$

or, since $\frac{1}{2}(e^{ix} + e^{-ix})$ is the expression for the hyperbolic cosine in which x is replaced by ix, again by definition,

$$\cosh ix = \cos x \qquad (5\cdot12\cdot4)$$

Subtracting Eq. (5·12·2) from Eq. (5·12·1) and dividing by 2, we find that

$$\tfrac{1}{2}(e^{ix} - e^{-ix}) = i \sin x \qquad (c)$$

or, remembering the definition of $\sinh x$,

$$\sinh ix = i \sin x \tag{5·12·5}$$

which defines the imaginary hyperbolic sine.

Changing x into ix on both sides of Eqs. (5·12·4) and (5·12·5), we obtain the definitions of the cosine and sine functions in the imaginary field.

$$\cos ix = \cosh i(ix) = \cosh(-x) = \cosh x \tag{5·12·6}$$

$$\begin{aligned} \sin ix &= \frac{1}{i} \sinh i(ix) = \frac{1}{i} \sinh(-x) = -\frac{1}{i} \sinh x \\ &= i \sinh x \end{aligned} \tag{5·12·7}$$

The cosine and sine of a complex variable can now be easily defined by means of the trigonometric formulas for the sum of two angles,

$$\begin{aligned} \cos(x + iy) &= \cos x \cos iy - \sin x \sin iy \\ &= \cos x \cosh y - i \sin x \sinh y \end{aligned} \tag{5·12·8}$$

$$\begin{aligned} \sin(x + iy) &= \sin x \cos iy + \cos x \sin iy \\ &= \sin x \cosh y + i \cos x \sinh y \end{aligned} \tag{5·12·9}$$

and, similarly, the hyperbolic cosine and hyperbolic sine of a complex variable become

$$\begin{aligned} \cosh(x + iy) &= \cosh x \cosh iy + \sinh x \sinh iy \\ &= \cosh x \cos y + i \sinh x \sin y \end{aligned} \tag{5·12·10}$$

$$\begin{aligned} \sinh(x + iy) &= \sinh x \cosh iy + \cosh x \sinh iy \\ &= \sinh x \cos y + i \cosh x \sin y \end{aligned} \tag{5·12·11}$$

A remarkable formula can be derived from Eq. (5·12·1) by making $x = k2\pi$ ($k = 0, \pm 1, \pm 2, \pm 3, \ldots$)

$$e^{k2\pi i} = \cos k2\pi + i \sin k2\pi$$

or

$$e^{k2\pi i} = 1 \tag{5·12·12}$$

A naïve reader, remembering that $e^0 = 1$ and comparing this definition with Eq. (5·12·12), may erroneously believe that $k2\pi i = 0$, whatever k. This is not so for the simple reason that $e^0 = 1$ is a definition valid in the field of real numbers, while Eq. (5·12·12) holds in the field of imaginaries. It goes without saying that, for $k = 0$, Eq. (5·12·12) is identical with $e^0 = 1$.

The reader may enjoy reviewing at this point how his knowledge of mathematics grew by stages during his education. He first met in arithmetic the natural numbers k with unit one; was later introduced to geometry and found the number π, ratio of the circumference to the

diameter of the circle; met the imaginary unit i in algebra; and learned the concept of limit and computed the irrational e in calculus. Equation (5·12·12) shows a simple relation between all these concepts and proves, if this needs proof, that the field of mathematics is beautifully integrated.

By means of Eq. (5·12·1) a complex number can be written in its *exponential form*. In Sec. 1·2 the complex number $z = x + iy$ was written in trigonometric form as

$$z = r(\cos\theta + i\sin\theta) \tag{d}$$

where

$$r = \sqrt{x^2 + y^2} \qquad \cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r} \tag{e}$$

But, since, by Eq. (5·12·1), $\cos\theta + i\sin\theta = e^{i\theta}$, z may also be written as

$$z = re^{i\theta} \tag{5·12·13}$$

It must be noticed here that the second and third of Eqs. (e) do not define a single value for the angle θ since $\cos(\theta + k2\pi) = \cos\theta$ and $\sin(\theta + 2k\pi) = \sin\theta$, provided that k be an integer. Hence a more general expression for the exponential form of z is

$$z = re^{i(\theta+k2\pi)} \tag{5·12·14}$$

By means of this last equation we can now define the logarithmic function of a complex variable. Applying formally the rule for the logarithm of the product of two numbers, we find that

$$\log_e z = \log_e r + i(\theta + k2\pi) \tag{5·12·15}$$

Equation (5·12·15) shows that the logarithm of a complex number is another complex number with real part $\log_e r$ and imaginary part $\theta + k2\pi$. Since k can be *any* integer, the $\log_e z$ is a *multivalued* function. In other words, there are infinite values of the logarithm of a complex number.

Taking $k = 0$ and limiting θ between $-\pi$ and $+\pi$ (*principal branch* of the logarithmic function), we obtain the single-valued complex logarithm.

$$\log_e z = \log_e r + i\theta \qquad (-\pi < \theta < \pi) \tag{5-12-16}$$

The elementary functions of a complex variable, as thus defined, reduce to the corresponding functions of a real variable whenever $z = x$. They have interesting properties and are usefully employed in many fields of engineering.

Problems

1. Evaluate the following polynomials at the given values of x by synthetic substitution:

(a) $3x^2 + 2x + 4$ $\quad x = 2, -2$
(b) $4x^3 - 2x^2 + x - 1$ $\quad x = -4, 2.5$
(c) $6x^4 + 2x^2 - 5$ $\quad x = 3, -1$
(d) $2.1x^3 - 6.4x^2 - 3.2x + 4.7$ $\quad x = 0.10, -0.75$
(e) $3.7x^4 + 2.4x^2 - 1.5x - 4.2$ $\quad x = 1.4, 0$
(f) $32x^6 - 16x^4 + 12x^3 - 5x + 1$ $\quad x = 2, -1$

2. Expand into partial fractions the following algebraic fractions:

(a) $\dfrac{s-2}{(s+1)(s-1)(s-3)}$ $\quad$ (b) $\dfrac{s^2-1}{s(s-2)(s+4)}$

(c) $\dfrac{s^3+s}{2s^2-1}$ $\quad$ (d) $\dfrac{s+1}{(2s^2+1)(s-1)}$

(e) $\dfrac{s^2+1}{s^3+s+2}$ $\quad$ (f) $\dfrac{s^2+1}{s^3+3s}$

(g) $\dfrac{1}{(s+2)(s^2+2s+1)}$ $\quad$ (h) $\dfrac{2(s+2)}{s^2(s+1)(s-1)}$

(i) $\dfrac{3}{(s^2+1)(s^2-2s-1)}$ $\quad$ (j) $\dfrac{15s^2+6s+33}{(s-1)^2(s+2)^2}$

3. Show on a unit circle the segments whose lengths measure the secant, cosecant, and cotangent, respectively.

4. Establish geometrically the following identities (see Fig. 5·12):

(a) $\sin^2 \alpha + \cos^2 \alpha = 1$
(b) $\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$
(c) $\sin 2\alpha = 2 \sin \alpha \cos \alpha$

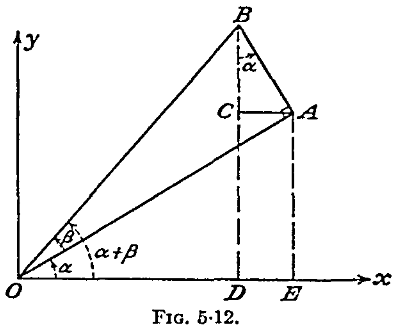


FIG. 5·12.

5. Prove that

(a) $\cos 5x = 16 \cos^5 x - 20 \cos^3 x + 5 \cos x$
(b) $\sin 4x = 8 \cos^3 x \sin x - 4 \cos x \sin x$
(c) $\sin x + \sin y = 2 \sin \dfrac{x+y}{2} \cos \dfrac{x-y}{2}$
(d) $\tan x - \tan y = \dfrac{\sin (x-y)}{\cos x \cos y}$

6. Evaluate the following

(a) cos 2.7
(b) sin (−4.2)
(c) tan 14.2
(d) sec 0.17
(e) cot (−2.1)
(f) csc 7.12

7. Write the following functions in terms of θ only:

(a) $\sin(\pi + \theta)$
(b) $\cos(\theta - 180°)$
(c) $\tan\left(\frac{3\pi}{2} - \theta\right)$
(d) $\cot\left(\frac{11\pi}{2} + \theta\right)$
(e) $\cos(90° - \theta)$
(f) $\tan\left(\frac{\pi}{2} - \theta\right)$
(g) $\sec\left(-\theta - \frac{3\pi}{2}\right)$
(h) $\sin(\theta - 450°)$
(i) $\sin\left(\frac{\pi}{2} + \theta\right)$
(j) $\cos(90° + \theta)$

8. Compute the derivatives of the following functions at the given value of x:

(a) $\sin x^2$ $\quad x = 0.5$
(b) $\cos\sqrt{x}$ $\quad x = \left(\frac{\pi}{2}\right)^2$
(c) $\tan x^{2/3}$ $\quad x = 1$
(d) $\sec 2x$ $\quad x = \frac{1}{4}$
(e) $\cot(x^2 - 2)$ $\quad x = 2$
(f) $\csc\sqrt{x+1}$ $\quad x = \frac{1}{2}$

9. State whether the following functions are even, odd, or neither:

(a) $\cos\sqrt{x}$
(b) $\cot x^2$
(c) $\sin x^3$
(d) $\csc(3x - 1)$
(e) $\sin x \cos x$
(f) $\cos^2 x - \sin^2 x$
(g) $\tan x^4$
(h) $\cot(\sin x)$

10. Verify on a unit circle that

	n odd	n even
$\sin n\pi$	0	0
$\cos n\pi$	-1	$+1$
$\sin\frac{n\pi}{2}$	$(-1)^{(n+1)/2}$	0
$\sin\frac{n\pi}{2}$	0	$(-1)^{n/2}$

11. Prove that, in a triangle of sides a, b, and c,

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

where α is the angle opposite a (*law of cosines*).

12. Prove that, in a triangle of sides a, b, and c,

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

where α, β, γ are the angles opposite a, b, and c, respectively (*law of sines*).

13. If a, b, and c are the 3 sides of a triangle and $s = \frac{1}{2}(a + b + c)$ show that

(*a*) the area A of the triangle is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

(*b*) $$\sin \frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

where α is the angle opposite a.

14. Evaluate the following functions (principal values) and their derivatives at the given values of x:

(*a*) $\arcsin x^2$	$x = 0.9$
(*b*) $\arccos \sqrt{x}$	$x = \frac{\pi}{4}$
(*c*) $\arctan x^3$	$x = 2$
(*d*) $\operatorname{arccot} (x - 1)$	$x = 1$
(*e*) $\operatorname{arcsec} \frac{2}{3}x^2$	$x = 1.5$
(*f*) $\arctan (x^4 + 2x^2)$	$x = 1.5$

15. State whether the principal values of the functions $\arcsin x$, $\arccos x$, $\arctan x$ and $\operatorname{arcsec} x$ are even, odd, or neither.

16. Using logarithms, calculate the following to 4 significant figures:

(*a*) $\sqrt[3]{0.046}$

(*b*) $(5.762)^5$

(*c*) $\dfrac{0.0193(84.16)}{\sqrt[3]{0.938}}$

(*d*) $\dfrac{(8.634)^3 \sqrt[5]{1.482}}{93.64}$

(*e*) $(5.4)^{\pi^2}$

(*f*) $e^{-0.71431}$

17. Simplify the following expressions:

(*a*) $e^{\ln x}$

(*b*) $e^{\frac{1}{2} \ln x}$

(*c*) $e^{\frac{2}{3} \ln (3x/2)}$

(*d*) $e^{-2 \ln x}$

(*e*) $e^{\ln x + \ln y}$

(*f*) $e^{2 \ln x - \ln y}$

(*g*) $e^{\ln (x/y)}$

(*h*) $e^{-y \ln x}$

18. Given: $e = 2.718$, $e^{0.1} = 1.105$, and $e^{0.01} = 1.010$, evaluate the following to 3 significant figures:

(*a*) $e^{2.11}$

(*b*) $e^{-1.22}$

(*c*) e^{π}

(*d*) $\sinh 1.1$

(*e*) $\cosh 2.15$

(*f*) $\tanh 2.92$

(*g*) $\coth (-1.21)$

(*h*) $\sinh (-2.12)$

19. For what values of x is the difference $\cosh x - \sinh x$

(*a*) less than 0.01

(*b*) less than 0.1 per cent of $\cosh x$

20. Prove that

$$A \cos (90° - x) + B \cos (90° + x) = (A - B) \sin x$$

21. Prove that

(a) $\cosh^2 x - \sinh^2 x = 1$
(b) $\sinh (x - y) = \sinh x \cosh y - \cosh x \sinh y$
(c) $\coth^2 x = \operatorname{csch}^2 x + 1$
(d) $\operatorname{sech}^2 x = 1 - \tanh^2 x$

22. Prove that

(a) $\sinh^{-1} x = \ln (x + \sqrt{x^2 + 1})$
(b) $\cosh^{-1} x = \ln (x + \sqrt{x^2 - 1})$

23. Show that the following relations exist between the circular and hyperbolic functions:

(a) $i \sinh x = \sin ix$ (b) $\cosh x = \cos ix$
(c) $i \tanh x = \tan ix$ (d) $\sinh ix = i \sin x$
(e) $\cosh ix = \cos x$ (f) $\tanh ix = i \tan x$

24. Establish the addition formula

$$\arctan \theta - \arctan \varphi = \arctan \left(\frac{\theta - \varphi}{1 + \theta\varphi}\right)$$

25. Prove that

$$\sin^2 \theta \sec^2 \theta + \cos^2 \theta \csc^2 \theta \sec^2 \phi + \tan^2 \varphi + 2 = \sec^2 \theta + \csc^2 \theta \sec^2 \varphi$$

26. Prove that

$$\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi \cos^2 \theta + \sin^2 \theta = 1$$

27. Prove that

$$\sqrt{\frac{1 - \sin \theta}{1 + \sin \theta}} = \sec \theta - \tan \theta$$

28. An F.M. wave has the following form,

$$i = I(1 + m \sin \omega_a t) \sin \omega_b t$$

where m is the degree of modulation and ω_a and ω_b are angular frequencies. Express i as the sum of simple sine and cosine terms.

29. Two voltages are given by

$$e_1 = 40 \sin (\omega t + \varphi_1)$$
$$e_2 = 50 \sin (\omega t + \varphi_2)$$

Express their sum in the form

$$e = A \sin (\omega t + \varphi)$$

30. Show that, for Ω real and $x > 1$,

$$S = \ln \left(\frac{x - \Omega^2}{1 - \Omega^2 x^2}\right) - T \ln x$$

where $\Omega = \sqrt{(T - 1)/(T + 1)}$, is always greater than zero.

31. Reduce the integral

$$t = \int_0^{\theta} \frac{d\theta}{\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}} \qquad (\theta < \theta_0)$$

to the integral

$$t = \sqrt{\frac{l}{g}} \int_0^{w} \frac{dw}{\sqrt{1 - \sin^2\left(\frac{\theta_0}{2}\right)\sin^2 w}}$$

by the transformation

$$\sin w = \frac{\sin\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta_0}{2}\right)}$$

The integral gives the time t versus the angular deflection θ of a pendulum released from a deflection θ_0.

32. Show that, if

$$\ln_{3b} a + \ln_b a = 2 \ln_{3b} a \ln_b a$$

then $a = b\sqrt{3}$.

33. The open-circuit voltage of a loop antenna is given by

$$V = E\cos\theta_0\cos\varphi_0 \left\{ \int_{+a}^{-a} e^{j(\omega t - \beta x)}\,dx + \int_{-a}^{+a} e^{j[\omega t - \beta(x + 2h)]}\,dx \right\}$$
$$- E\sin\theta_0 \left\{ \int_0^{2h} e^{j[\omega t - \beta(-a - x)]}\,dx + \int_{2h}^{0} e^{j[\omega t - \beta(a + x)]}\,dx \right\}$$

where $\beta = 2\pi/\lambda$, and a, h are dimensions of the loop.

If λ (wave length) $\gg a,h$, show that

$$V = -j\beta EA(\cos\varphi_0 \cdot \cos\theta_0 + \sin\theta_0)e^{j\omega t}$$

where $A = 4ah$ = area of the loop.

34. Figure 5-13 shows the bending-moment diagram for a beam on two simple supports. Draw the shear and the load diagrams.

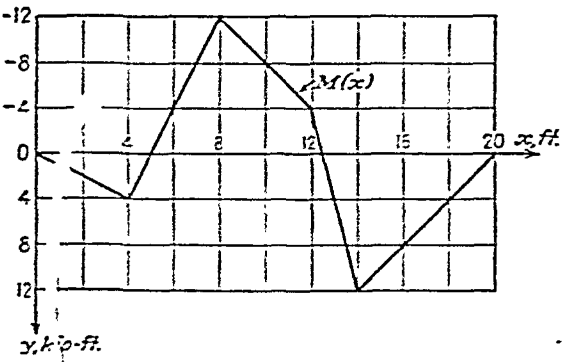


FIG. 5-13.

35. Show that, if

$$\cos x + \cos y + \cos z = \sin x + \sin y + \sin z = 0$$

then

$$\sin^2 x + \sin^2 y + \sin^2 z = \cos^2 x + \cos^2 y + \cos^2 z = 3/2$$

Hint: Use exponential forms of sine and cosine.

36. Prove the validity of the construction of Pascal's triangle, *i.e.*, show that

$$C_{m,n} + C_{m+1,n} = C_{m+1,n+1}$$

where

$$C_{m,n} = \frac{n!}{(n-m)!m!}$$

37. Solve the following system for x,y,z, and t in terms of a, $\csc\alpha$, and $\cot\alpha$ *only:*

$$\begin{aligned} 6x + 6y &= -a \\ x\cos\alpha + y\cos 3\alpha + z\sin\alpha + t\sin 3\alpha &= 0 \\ z + 3t &= 0 \\ 2x\sin\alpha + 6y\sin 3\alpha - 2z\cos\alpha - 6t\cos 3\alpha &= 0 \end{aligned}$$

38. Establish the identity

$$\left(\frac{dx}{dy}\right)^2 \frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 \frac{d^3x}{dy^3} + 3\frac{d^2y}{dx^2}\frac{d^2x}{dy^2} = 0$$

Hint: $dy/dx = 1/(dx/dy)$.

39. If $y = \cos x + x\sin x$, show that

$$\frac{d^ny}{dx^n} = x\sin\left(\frac{n}{2}\pi + x\right) + (n-1)\sin\left[(n-1)\frac{\pi}{2} + x\right]$$

40. The elevation of a water tower is α from a point A and β from a point B a distance d from A away from the tower. Prove that the height of the tower is given by

$$h = \frac{d}{\cot\beta - \cot\alpha}$$

41. Show that the radius of a circle circumscribed about a triangle of sides a,b, and c is given by

$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

where $s = \frac{1}{2}(a + b + c)$.

42. Two motorboats start at the same time from the same dock. One travels in the direction 75° east of south at 40 mph. In what direction must the other travel at 50 mph in order to be due north of the first?

43. Two beach lights are due east of a dock at distances of 200 yd and 2000 yd. An observer on a boat due south of the dock finds at one point that the difference in the directions of the lights is 45 deg. After proceeding toward the dock a certain distance, he again finds the difference in direction of the lights to be 45 deg. What is the distance traveled between the 2 points of observation?

44. A strip of 1000 sq yd is sold from a triangular field, whose sides are 120, 90, and 100 yd. The strip is of uniform width h and has one of its sides parallel to the longest side of the field. Find the width of the strip. (See Fig. 5·14.)

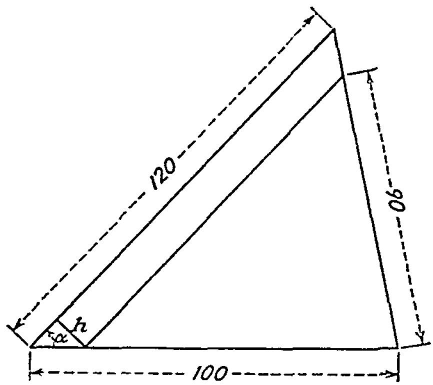


FIG. 5·14.

45. A triangular field has 2 sides 2 miles and 1.75 miles long, while the angle included between them is 42°50′. Compute the third side of the field and the other 2 angles.

46. A triangular field has 2 sides 475 ft and 252 ft long. The angle opposite the longer side is 22°10′. Compute the area of the field.

47. The population of a city grows at the rate of 3.5 per cent per year. What will the population be 20 years hence if the present population is 100,000?

48. The present population of New York City is 6,000,000. Its birth rate is 3 per cent per year; its death rate, 2 per cent per year. How many people will be born in New York in the next 50 years if the rates remain unchanged?

49. The electric lights of a city go out at the rate of 0.5 per cent per month. Because of shortage of bulbs only 10,000 can be replaced each month. If the initial number of lights is 10,000,000, in how many months will the supply be sufficient to replace the bulbs going out each month?

50. Expand by the binomial theorem and exhibit the first 3 terms of the expansion of the following functions:

(a) $(1 + x)^{1/2}$ (b) $(x^2 - y^2)^{-1/2}$
(c) $(a - bx)^{1/3}$ (d) $(1 - x^2)^{1/4}$
(e) $(1 - x)^n$ (f) $(a^3 + x^3)^{-1/3}$

51. Evaluate the following to 3 significant figures by means of the binomial expansion:

(a) $\sqrt{17.5}$ (b) $\sqrt[3]{30}$
(c) $\dfrac{1}{\sqrt{83}}$ (d) $(1.12)^6$
(e) $(0.93)^{-4}$ (f) $\sqrt[4]{262}$

52. Evaluate the following integrals by expanding the integrand by the binomial theorem up to the third term of the expansion:

(a) $\int_0^1 \sqrt{1 - x^2}\, dx$ (b) $\int_0^{0.5} \frac{dx}{\sqrt{1 - x^2}}$

(c) $\int_0^1 \sqrt[4]{1 + x}\, dx$ (d) $\int_0^{1.5} \sqrt[3]{8 + x}\, dx$

(e) $\int_1^0 (4 - x)^{-1/2}\, dx$ (f) $\int_1^2 \frac{dx}{\sqrt{4 + x^2}}$

53. Expand the following functions into Maclaurin's series. Exhibit the first 4 (nonzero) terms of the expansion.

(a) $\sinh x$ (b) $\cosh x$
(c) e^{x^2} (d) e^{-x^2}
(e) $\tan x$ (f) $\arcsin x$
(g) $\ln (1 + x)$ (h) $\sqrt{1 - x^2}$

54. Evaluate the upper bound of the error after the third (nonzero) term of the expansions of Prob. 53*a*, *b*, *c*, and *g*.

55. Expand by Taylor's series the following functions about the indicated points up to the third nonzero term:

(a) $\sin x$ $x_0 = \frac{\pi}{2}$
(b) e^x $x_0 = 1$
(c) $\cos 3x$ $x_0 = 15°$
(d) $\ln x^2$ $x_0 = 2$
(e) $\cosh x$ $x_0 = 3$
(f) $\tan x$ $x_0 = \frac{\pi}{4}$

56. Evaluate the upper bound of the error in the expansions of Prob. 55*a* to *e* after the third nonzero term.

57. Evaluate the following to 3 significant figures:

(a) $e^{2.9}$ (b) $\sin 0.0175$
(c) $\ln 3$ (d) $\arcsin 0.9$
(e) $\sqrt{e}$ (f) $\cosh 3.1$

58. Remembering that $\int dx/(1 + x^2) = \arctan x + c$, obtain a series expansion for $\pi/4$.

59. Express the series

$$x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots$$

in finite terms. *Hint:* Differentiate the series, identify the corresponding function, and integrate.

60. A function $y(x)$ has the following properties:

(a) $y(0) = 1$ (b) $y'(0) = a$ (c) $y''(x) = a^2y(x)$.

Find $y(x)$ *Hint:* Differentiate *c* repeatedly, and use the results to form a Maclaurin's series.

61. Evaluate the elliptic integral

$$F(x,k) = \int_0^x \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}$$

for $k = 0.5$ and $x = \pi/2$ by expansion of the integrand into a series up to its third term.

62. Evaluate the following integrals to 3 significant figures by their series expansion:

(a) $\int_0^1 \tanh x \, dx$ (b) $\int_1^{1.2} \frac{\sqrt{x^2+1}}{x} \, dx$

(c) $\int_0^1 \sqrt{x} \sin x \, dx$ (d) $\int_1^{1.1} \frac{e^x}{x} \, dx$

(e) $\int_0^{0.2} e^{-x^2/2} \, dx$ (f) $\int_0^{0.5} e^{\sqrt{x}} \, dx$

63. A monument to the soldiers of the Civil War has a group of 4 mutually tangent cannon balls all 10 in. in diameter. How high is the group?

64. The mast of a ship is h ft above sea level. Assuming h to be very small in comparison with the radius R of the earth, prove that the angle θ under which the lookout sees the horizon is given by $\theta = \sqrt{2h/R}$. (See Fig. 5·15.)

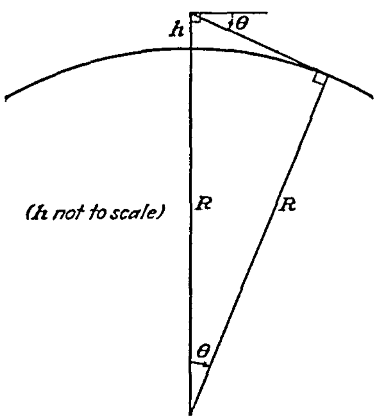


FIG. 5·15.

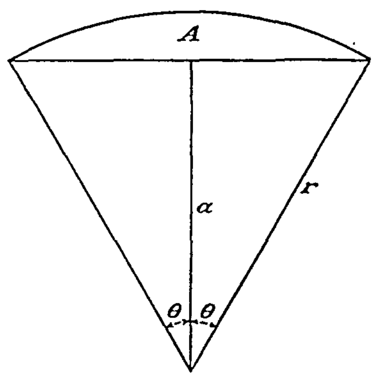


FIG. 5·16.

65. Obtain an approximate formula (in terms of r and θ) for the difference in length between the arc of a circle of radius r subtending an angle 2θ and the chord of the same arc, by using the first 3 terms of the expansion of $\sin \theta$. What is the maximum value of θ for this difference to be less than 1 per cent of the arc?

66. Derive an approximate formula for the area A of a segment of a circle (see Fig. 5·16) in terms of its apothem a and radius -

(a) if $a \ll r$
(b) if $a \doteq r$

67. A crank arm AB of length r rotates with a constant angular velocity ω. The crosshead C of the connecting rod l moves along the x axis. Obtain an approximate

expression for the displacement of the crosshead as a function of ω, r, and l and, by differentiation, an approximate expression for its velocity. (See Fig. 5·17.) Assume $r/l \ll 1$.

Note: $\theta = \omega t$.

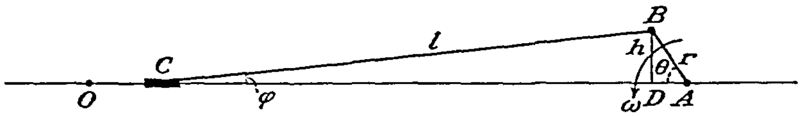


FIG. 5·17.

68. In elementary beam theory the curvature $y''/[1 + (y')^2]^{3/2}$ is approximated by y''. What would be a better approximation? Compare this approximation with y'' and with the exact value of the curvature at the point $x = 0.1$ of the curve $y = x^3/3$.

69. A continuous curve has its first 5 derivatives equal to zero at a point (x_0,y_0). What order parabola should you use to approximate the curve in the neighborhood of (x_0,y_0)? Write the equation of the parabola.

70. Test whether the following series are convergent or divergent, after writing the nth term for each series:

(a) $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$

(b) $\frac{1}{5} + \frac{1}{2\cdot 5} + \frac{1}{3\cdot 5} + \cdots$

(c) $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \frac{5}{2^5} + \frac{6}{2^5} + \cdots$

(d) $\frac{2!}{100^2} + \frac{3!}{100^3} + \frac{4!}{100^4} + \cdots$

(e) $1 + \frac{3}{7} + \frac{5}{7^2} + \frac{7}{7^3} + \frac{9}{7^4} + \cdots$

(f) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \cdots$

(g) $1 + \frac{1}{\sqrt{2^5}} + \frac{1}{\sqrt{3^5}} + \frac{1}{\sqrt{4^5}} + \cdots$

(h) $\frac{2}{2 \times 3 \times 4} + \frac{4}{3 \times 4 \times 5} + \frac{6}{4 \times 5 \times 6} + \cdots$

(i) $\frac{1}{2+1} + \frac{1}{4+2} + \frac{1}{8+3} + \cdots$

(j) $\frac{1}{\log 2} + \frac{1}{\log 4} + \frac{1}{\log 6} + \cdots$

71. Evaluate the following limit:

$$\lim_{n\to\infty} \left| \ln (1 + x) - \sum_{k=1}^{n} (-1)^{k+1} \frac{x^k}{k} \right|$$

72. Determine whether the series whose nth term is given below are convergent or divergent:

(a) $\frac{1}{4^n - 2}$

(b) $\frac{n!}{75^{n-1}}$

(c) $\frac{4^n}{n}$

(d) $\frac{n!}{7!(n-7)!}$

(e) $(\ln 2)^{-n}$ (f) $(\ln 3)^{-n}$

(g) $(-1)^n \dfrac{n(n-1)}{(n+1)^2}$

73. Write the nth term and determine the interval of convergence of the following power series:

(a) $1 + x^2 + x^4 + \cdots$ (b) $1 + \dfrac{x^3}{2} + \dfrac{x^5}{3} + \dfrac{x^7}{4} + \cdots$

(c) $1 + \dfrac{x^2}{3^2} + \dfrac{x^4}{5^2} + \dfrac{x^6}{7^2} + \cdots$ (d) $1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \cdots$

(e) $x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \cdots$ (f) $x + \dfrac{x^2}{\sqrt{2}} + \dfrac{x^3}{\sqrt{3}} + \cdots$

(g) $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$

(h) $1 + \dfrac{3x}{2} + \dfrac{3^2x^2}{5} + \dfrac{3^3x^3}{10} + \dfrac{3^4x^4}{17} + \dfrac{3^5x^5}{26} + \cdots$

(i) $\ln x + \ln x^2 + \ln x^3$ (j) $\ln x + \ln 2x + \ln 3x$

(k) $1 + (x-1) + \dfrac{(x-1)^2}{3!} + \dfrac{(x-1)^3}{5!} + \cdots$

(l) $\dfrac{1}{2\cdot 3} + \dfrac{(x-3)}{3\cdot 4} + \dfrac{(x-3)^2}{4\cdot 5} + \dfrac{(x-3)^3}{5\cdot 6}$

74. Find the sum of the numbers in the nth group.

(1) $1 = 1$
(2) $2 + 3 = 5$
(3) $4 + 5 + 6 = 15$
(4) $7 + 8 + 9 + 10 = 34$
(5) $11 + 12 + 13 + 14 + 15 = 65$

75. Write the following complex numbers in the form $a + bi$:

(a) $4z - 2z + 3 \qquad z = -2 + i$

(b) $\dfrac{3z^2 - 4}{z^4 + 2z^2 + 1} \qquad z = 2i$

(c) $e^{(-\pi/4)i}$ (d) $e^{2+(i\pi/6)}$

(e) $\sinh (4i)$ (f) $\cosh (-2i)$

(g) $\sin (\pi i)$ (h) $\tan \dfrac{\pi}{2} i$

(i) $\cos \left(\dfrac{\pi}{2} + \pi i\right)$ (j) $\sinh (-\pi + \pi i)$

(k) $4e^{3\frac{1}{2}\pi i}$ (l) $\ln (4 - 3i)$

76. Find the value of the following limits when they exist:

(a) $\lim\limits_{z\to\infty} e^{iz} \qquad z = a + 0i \qquad z = 0 + bi$

(b) $\lim\limits_{z\to\infty} e^{-iz} \qquad z = 0 + bi \qquad z = a + 0i$

77. Compute the moduli of the following functions:

(*a*) e^{ix} (*b*) e^{-ix}
(*c*) $\sinh ix$ (*d*) $\tanh ix$
(*e*) $\cos \frac{\pi}{4}(1 + i)$ (*f*) $\ln (4 + 3i)$

78. Solve the following equations for the unknown x:

(*a*) $\sin x = 2$ (*b*) $\cosh x = 0$
(*c*) $\sinh nx = i$ (*d*) $e^{nix} = i$

79. Solve the following equations for x and y:

(*a*) $(x - yi)^2 = 3 + 2i$ (*b*) $(x + yi)^2 = 7$
(*c*) $(x + yi)^2 = 4 + 3i$ (*d*) $\dfrac{1}{x - yi} = 1 + 2i$

80. Show that if

$$A = 1 + \omega^p + \omega^{2p} + \cdots + \omega^{(r-1)p}$$

where $\omega = e^{2\pi i/r}$ and p and r are integers, then

$$A = 0 \text{ if } p \text{ is not divisible by } r$$
$$A = r \text{ if } p \text{ is divisible by } r$$

81. Establish the following formulas, using complex notation:

(*a*) $\sin (x + y) = \sin x \cos y + \sin y \cos x$
(*b*) $\cos (x - y) = \cos x \cos y + \sin x \sin y$

Hint: $e^{ix} = \cos x + i \sin x$.

CHAPTER VI

FOURIER SERIES EXPANSION AND HARMONIC ANALYSIS

6·1 Introduction

Many of the quantities appearing in engineering problems have the property known as *periodicity*. A function $f(x)$ is called periodic when

$$f(x + k) = f(x) \tag{6·1·1}$$

where k is a constant called the *period*. The graph of a periodic function repeats identically the graph corresponding to any interval x, $x + k$ for the variable x.

The pressure in the cylinder of a reciprocating engine, the sea tide at a given point of the earth, the voltage, current, or flux density in an electric motor—all are examples of periodic functions of time. These are often known experimentally, but it is valuable and sometimes necessary to know their mathematical expression. Thus, once the tide function is known, tides can be easily predicted and tide tables can be published in advance, to the advantage of navigation. The important field of applied mathematics concerned with the analysis of experimental periodic functions is called *harmonic analysis*.

In Chap. V certain functions have been expanded into power series by means of Taylor's theorem; let us consider the geometric meaning of this expansion. If we let

$$y_1 = a_0 + a_1x, \qquad y_2 = a_2x^2, \qquad y_3 = a_3x^3, \qquad \cdots$$

Taylor's expansion of a function $y(x)$ can be written as

$$y = y_1 + y_2 + y_3 + \cdots + y_n + \cdots$$

and shows that the graph of a function y can be approximated by the sum of the graphs of a straight line y_1, a quadratic parabola y_2, a cubic parabola y_3, etc. The graphs of other types of functions might be used to approximate the graph of a given function; but when the function under study is periodic, it would seem logical to approximate it by means of periodic functions and more specifically by means of the simplest and best known periodic functions, cosine and sine. The expansion of a function known analytically into a series of sines and cosines is the concern of *Fourier analysis*, one of the fundamental branches of applied mathematics, whose full appreciation requires an elementary knowledge of the differential equations of physics.

In what follows we shall first cover the fundamentals of Fourier analysis and then review the elementary techniques of harmonic analysis.

6·2 Fourier Expansion in $-\pi,+\pi$

a. In a paper presented to the Paris Academy in 1807 (since become one of the most famous writings in the history of mathematics, both because of its importance and because of its rejection by a jury due to "lack of rigorous proofs") the French physicist Fourier, in order to solve a problem in heat flow, assumed that a function $y = f(x)$ could be expanded into a series of the type

$$f(x) = \tfrac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$

that is, let

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \tag{6·2·1}$$

where the a_n and b_n are constants.

The series (6·2·1) is now called the *Fourier series* of $f(x)$. Its terms are sine and cosine functions, which make n complete oscillations ($n = 1$, 2, 3, . . .) in the interval $-\pi,+\pi$ and hence have all the common period 2π. The right-hand member of Eq. (6·2·1) is therefore a periodic function of period 2π; and if $f(x)$ has this same period, the problem of the Fourier expansion of $f(x)$ consists in the determination of the coefficients $a_0,a_1,a_2, \ldots ,b_1,b_2,b_3, \ldots$ of Eq. (6·2·1).

This determination is based on the following fundamental properties of the trigonometric functions, called *orthogonality conditions:*

$$\left.\begin{aligned} &\int_{-\pi}^{+\pi} \cos mx \cos nx\, dx = \int_{-\pi}^{+\pi} \sin mx \sin nx\, dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases} \\ &\int_{-\pi}^{+\pi} \cos mx \sin nx\, dx = 0 \qquad\qquad (m,n \text{ integers}) \end{aligned}\right\} \tag{6·2·2}$$

To prove, for instance, the first of the orthogonality conditions for $m \neq n$, notice that

$$\cos mx \cos nx = \tfrac{1}{2} \cos (m + n)x + \tfrac{1}{2} \cos (m - n)x$$

and integrate between $-\pi$ and $+\pi$ to obtain

$$\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos (m+n)x\, dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos (m-n)x\, dx$$
$$= \frac{1}{2(m+n)} \sin (m+n)x\Big]_{-\pi}^{\pi} + \frac{1}{2(m-n)} \sin (m-n)x\Big]_{-\pi}^{\pi}$$
$$= \frac{1}{2(m+n)} (0 - 0) + \frac{1}{2(m-n)} (0 - 0) = 0$$

To prove the first condition for $m = n$, notice that, when $m = n$,

$$\cos mx \cos mx = \cos^2 mx = \tfrac{1}{2}(1 + \cos 2mx)$$

and integrate between $-\pi$ and $+\pi$ to obtain

$$\int_{-\pi}^{\pi} \cos^2 mx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} dx + \frac{1}{4m}\int_{-\pi}^{\pi} \cos 2mx\, d(2mx)$$
$$= \frac{1}{2}x\Big]_{-\pi}^{\pi} + \frac{1}{4m}\sin 2mx\Big]_{-\pi}^{\pi} = \frac{1}{2}(\pi + \pi) + \frac{1}{4m}(0 + 0) = \pi$$

The other orthogonality conditions are similarly proved.

Just as Maclaurin used differentiation to obtain the coefficients of a power-series expansion, Fourier used integration. To obtain the constant a_0 he multiplied both sides of Eq. (6·2·1) by $\cos 0x = 1$ and integrated between $-\pi$ and $+\pi$.

$$\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2}a_0\int_{-\pi}^{\pi} \cos 0x\, dx + \sum_{n=1}^{\infty} a_n \int_{-}^{} \cos 0x \cos nx\, dx$$
$$+ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \cos 0x \sin nx\, dx$$

By Eqs. (6·2·2) all the integrals under the series signs are equal to zero, since $m = 0$ and $n \neq 0$, and the equation is simplified to

$$\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2}a_0\int_{-\pi}^{\pi} \cos 0x\, dx = \frac{1}{2}a_0\int_{-\pi}^{\pi} dx = \frac{1}{2}a_0x\Big]_{-\pi}^{\pi} = a_0\pi$$

from which

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx \qquad (a)$$

Similarly, to obtain a *given* cosine coefficient $a_{\bar{n}}$, multiply both sides of Eq. (6·2·1) by $\cos \bar{n}x$ and integrate between $-\pi$ and $+\pi$.

$$\int_{-\pi}^{\pi} f(x)\cos \bar{n}x\, dx = \frac{1}{2}a_0\int_{-\pi}^{\pi} \cos \bar{n}x\, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos \bar{n}x \cos nx\, dx$$
$$+ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \cos \bar{n}x \sin nx\, dx$$

Owing to the orthogonality conditions (6·2·2), all the integrals in this equation are zero except the cosine integral for which the running sub-

script n equals the given integer $\bar{n}$; hence the equation reduces to

$$\int_{-\pi}^{\pi} f(x) \cos \bar{n}x \, dx = a_{\bar{n}} \int_{-\pi}^{\pi} \cos \bar{n}x \cos \bar{n}x \, dx$$

and, by the first orthogonality condition for $m = n$,

$$\int_{-\pi}^{\pi} f(x) \cos \bar{n}x \, dx = a_{\bar{n}}\pi$$

from which, reverting to the general subscript n,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \tag{6·2·3}$$

It must be noticed that, for $n = 0$, Eq. (6·2·3) gives the value (a) of the first constant a_0. The first term of the Fourier series is called $\frac{1}{2}a_0$ and not a_0 in order to avoid the use of a special formula for its computation.

To compute a *given* sine coefficient $b_{\bar{n}}$, multiply both sides of Eq. (6·2·1) by $\sin \bar{n}x$, and integrate between $-\pi$ and $+\pi$.

$$\int_{-\pi}^{\pi} f(x) \sin \bar{n}x \, dx = \frac{1}{2} a_0 \int_{-\pi}^{\pi} \sin \bar{n}x \, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \sin \bar{n}x \cos nx \, dx$$
$$+ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin \bar{n}x \sin nx \, dx$$

Owing to the orthogonality conditions (6·2·2), all the integrals in this equation are zero except the sine integral, for which the running subscript n equals $\bar{n}$; hence the equation simplifies to

$$\int_{-\pi}^{\pi} f(x) \sin \bar{n}x \, dx = b_{\bar{n}} \int_{-\pi}^{\pi} \sin \bar{n}x \sin \bar{n}x \, dx$$

and, by the second orthogonality condition for $m = n$,

$$\int_{-\pi}^{\pi} f(x) \sin \bar{n}x \, dx = b_{\bar{n}}\pi$$

from which, reverting to the general subscript n,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \tag{6·2·4}$$

The expansion (6·2·1), with the coefficients a_n and b_n given by Eqs. (6·2·3) and (6·2·4), respectively, is called the *Fourier expansion of* $f(x)$ *in the complete Fourier interval* $-\pi,+\pi$.

b. The expressions for the Fourier coefficients can be simplified when the function $f(x)$ is either even or odd.

Since a_n is given by a *definite* integral, the name of its variable of integration can be changed from x to z without altering the value of a_n (see Sec. 1·9a). Moreover, splitting the integral giving a_n into two separate integrals, we can write

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(z)\cos nz\,dz = \frac{1}{\pi}\int_{-\pi}^{0} f(z)\cos nz\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(z)\cos nz\,dz \qquad (b)$$

Let us now set in the first integral $z = -t$ and therefore

$$f(z) = f(-t) \qquad \cos nz = \cos n(-t) = \cos nt$$
$$dz = -dt \qquad z = -\pi,\ t = \pi \qquad z = 0,\ t = 0$$

obtaining

$$\frac{1}{\pi}\int_{-\pi}^{0} f(z)\cos nz\,dz = \frac{1}{\pi}\int_{\pi}^{0} f(-t)\cos nt(-dt) = \frac{1}{\pi}\int_{0}^{\pi} f(-t)\cos nt\,dt$$

Substituting this expression in Eq. (b) and changing the name of the variable of integration back to x in both integrals, we obtain

$$a_n = \frac{1}{\pi}\int_{0}^{\pi} f(-x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$
$$= \frac{1}{\pi}\int_{0}^{\pi}[f(x) + f(-x)]\cos nx\,dx \qquad (c)$$

Similarly, splitting the integral appearing in Eq. (6·2·4) into two integrals

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} f(z)\sin nz\,dz + \frac{1}{\pi}\int_{0}^{\pi} f(z)\sin nz\,dz \qquad (d)$$

and substituting in the first integral $z = -t$ and therefore

$$f(z) = f(-t) \qquad \sin z = \sin(-t) = -\sin t$$
$$dz = -dt \qquad z = -\pi,\ t = \pi \qquad z = 0,\ t = 0$$

we obtain

$$\frac{1}{\pi}\int_{-\pi}^{0} f(z)\sin nz\,dz = \frac{1}{\pi}\int_{\pi}^{0} f(-t)(-\sin t)(-dt)$$
$$= -\frac{1}{\pi}\int_{0}^{\pi} f(-t)\sin nt\,dt$$

and, changing the name of the variable of integration back to x in both integrals of Eq. (d),

$$b_n = -\frac{1}{\pi}\int_{0}^{\pi} f(-x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$
$$= \frac{1}{\pi}\int_{0}^{\pi}[f(x) - f(-x)]\sin nx\,dx \qquad (e)$$

When $f(x)$ is an *even* function, $f(x) = f(-x)$ and, by Eqs. (*c*) and (*e*),

$$a_n = \frac{2}{\pi}\int_0^\pi f(x) \cos nx\, dx \qquad b_n = 0 \quad [f(x) \text{ an even function}] \tag{6·2·5}$$

When $f(x)$ is an *odd* function, $f(x) = -f(-x)$ and, again by Eqs. (*c*) and (*e*),

$$a_n = 0 \qquad b_n = \frac{2}{\pi}\int_0^\pi f(x) \sin nx\, dx \quad [f(x) \text{ an odd function}] \tag{6·2·6}$$

Equations (6·2·5) and (6·2·6) give the Fourier coefficients of even and odd functions in the complete Fourier interval $-\pi, +\pi$.

c. It was noticed in Sec. 6·2*a* that the function $f(x)$ defined by the series (6·2·1) is necessarily a periodic function of period 2π; hence only periodic functions of period 2π can be expanded, if we wish the Fourier series to equal the given function $f(x)$ *for all values of* x. If, instead, the range of values with which we are concerned is limited to the Fourier interval $-\pi, +\pi$, we can expand into a Fourier series *any* function $f(x)$ by means of what is known as *periodic prolongation.*

If, for instance, we wish to expand the function $y = x$ in a Fourier series *in the interval* $-\pi, +\pi$, this means that we wish to find a Fourier series of which the sum equals x between $-\pi, +\pi$ but which is not necessarily equal to x outside this interval. To this end, *prolong* the graph of $y = x$ beyond $-\pi$ and $+\pi$ by repeating it identically in the intervals . . . ; -5π, -3π; -3π, $-\pi$; π, 3π; 3π, 5π; . . . , as shown in Fig. 6·1.

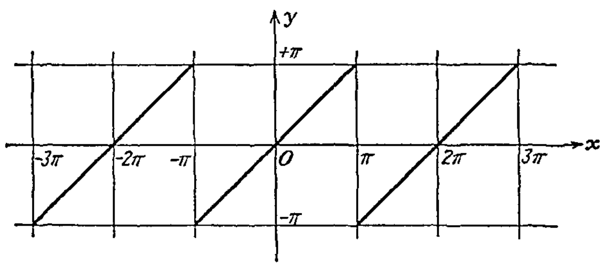


FIG. 6·1.

The new function, thus obtained by *periodic prolongation* of the function $y = x$ given in the interval $-\pi, +\pi$, is periodic of period 2π and may be expanded into a Fourier series, which will equal x between $-\pi$ and $+\pi$ but will be different from $y = x$ outside this interval.

Since $y = x$ is an odd function, its Fourier coefficients are given by Eq. (6·2·6),

$$a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^\pi f(x) \sin nx\, dx = \frac{2}{\pi}\int_0^\pi x \sin nx\, dx$$

$$= \frac{2}{\pi n}\int_0^\pi x\,(-d\cos nx) = \frac{2}{\pi n}\left\{-x\cos nx\Big]_0^\pi + \int_0^\pi \cos nx\,dx\right\}$$
$$= \frac{2}{\pi n}\left\{-\pi\cos n\pi + \frac{1}{n}\sin nx\Big]_0^\pi\right\} = -\frac{2}{n}\cos n\pi$$

or, since $\cos n\pi$ equals $+1$ for n even and -1 for n odd,

$$b_n = (-1)^{n-1}\frac{2}{n}$$

by means of which Eq. (6·2·1) gives

$$f(x) = x = 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin nx}{n} \qquad (-\pi < x < +\pi) \tag{6·2·7}$$

Figure 6·2 shows how the graphs, obtained by taking an increasing number of terms of Eq. (6·2·7), approach the graph of $y = x$ between $-\pi$ and $+\pi$.

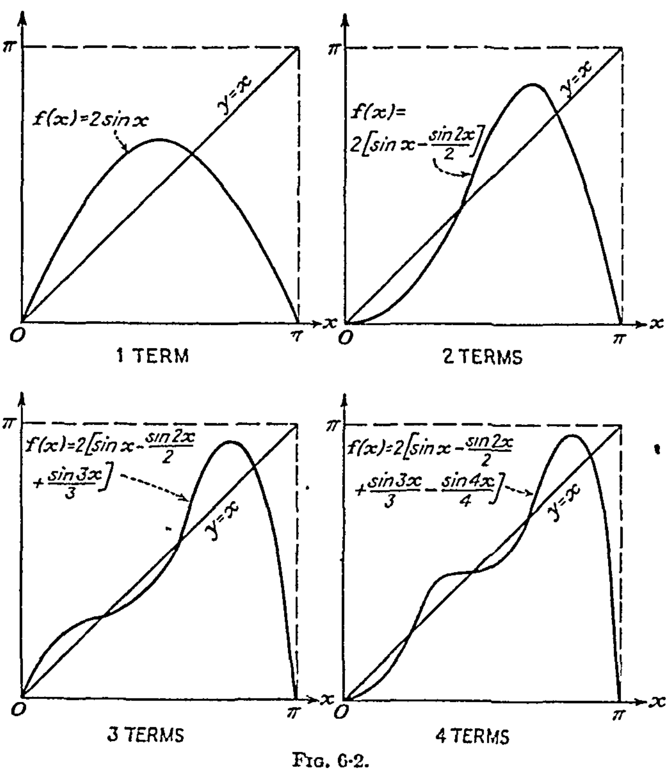


FIG. 6·2.

Since every term of Eq. (6·2·7) becomes zero at $x = k\pi$ ($k = \pm 1, \pm 2, \pm 3, \ldots$), the series equals zero at $x = k\pi$. It will be noticed that the periodically prolonged function $y = x$ is discontinuous at $x = k\pi$, since it jumps there from $+\pi$ to $-\pi$, and that the average of the two values to the right and left of the discontinuity is zero. It can be stated in general that *the Fourier series of a function will converge toward the value of the function at points where the function is continuous; toward the average of the two limits from left and right at points of discontinuity.*

d. Since a function does not have to be continuous in order to be Fourier expandable, we can expand into Fourier series functions that are defined by more than one mathematical expression. For instance, the function

$$y = \begin{cases} -1 & -\pi < x < 0 \\ +2 & 0 < x < +\pi \end{cases}$$

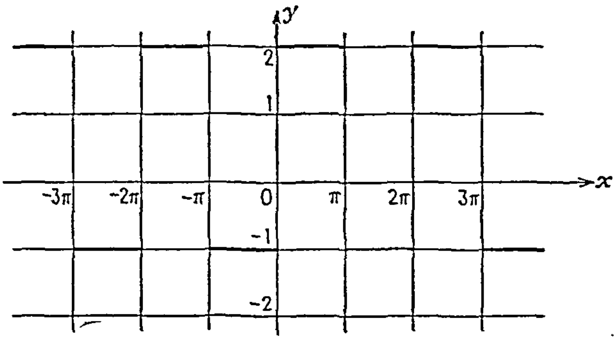


FIG. 6·3.

whose periodic prolongation is shown in Fig. 6·3, has the Fourier coefficients

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} (-1)\,dx + \frac{1}{\pi}\int_{0}^{\pi} (2)\,dx = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} (-1)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} (2)\cos nx\,dx$$

$$= -\frac{1}{\pi n}\sin nx\Big]_{-\pi}^{0} + \frac{2}{\pi n}\sin nx\Big]_{0}^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} (-1)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} (2)\sin nx\,dx$$

$$= \frac{1}{\pi n}\cos nx\Big]_{-\pi}^{0} - \frac{2}{\pi n}\cos nx\Big]_{0}^{\pi} = \frac{3}{\pi n}(1 - \cos n\pi) = \begin{cases} \dfrac{6}{n\pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$

and a Fourier expansion given by

$$f(x) = \frac{1}{2} + \frac{6}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin nx}{n} \qquad (-\pi < x < +\pi)$$

At $x = k\pi (k = 0, \pm 1, \pm 2, \cdots)$ the value of the series is $\frac{1}{2}$, average of the two values -1 and 2 to the left and right of the discontinuity.

6·3 Fourier Expansion in 0, π

When the range of values of x, in which we are interested, is limited to the *half Fourier interval* $0, \pi$, the periodic prolongation of the given function is not unique but can be performed in an infinite variety of ways. For instance, the function $y = x$ given in the interval $0, \pi$ can be prolonged in the other half interval $-\pi, 0$ as shown in Fig. 6·4*a*,

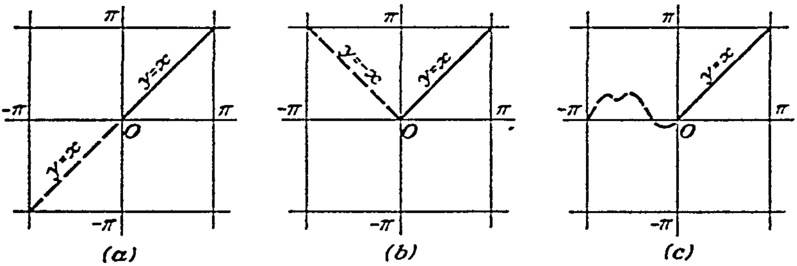


FIG. 6·4.

giving rise to an odd function, or as shown in Fig. 6·4*b*, giving rise to an even function, or as shown in Fig. 6·4*c*, giving rise to a function that is neither odd nor even. Whenever there is a free choice in the prolongation of y to the left of the origin, even or odd prolongations are to be preferred, since they save the computation of half the Fourier coefficients. It is thus seen that a function given in the *complete* Fourier interval has a unique expansion, while a function given in the *half* Fourier interval may have a sine, a cosine, and an infinity of sine and cosine expansions.

For instance, the function $y = x$, which had to be expanded into a sine series when given in $-\pi,\pi$, has instead the following cosine expansion when given in $0, \pi$:

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx = \frac{2}{\pi n} \left\{ x \sin nx \Big]_0^{\pi} - \int_0^{\pi} \sin nx \, dx \right\}$$

$$= \frac{2}{\pi n^2} \cos nx \Big]_0^{\pi} = \frac{2}{\pi n^2} (\cos n\pi - 1) = \begin{cases} 0 & \text{for } n \text{ even} \\ -\dfrac{4}{\pi n^2} & \text{for } n \text{ odd} \end{cases}$$

$$a_0 = \frac{2}{\pi}\int_0^\pi x\,dx = \frac{2}{\pi}\frac{x^2}{2}\Big]_0^\pi = \pi$$

$$f(x) = x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{\cos nx}{n^2} \quad (0 < x < \pi) \qquad (6\cdot3\cdot1)$$

Equation (6·3·1) converges more rapidly than Eq. (6·2·7), since in Eq. (6·3·1) the increasing integers n appear squared at the denominators of the successive terms of the series.

The even- and odd-function coefficients (6·2·5) and (6·2·6) can thus be used for the expansion in terms of sines only or of cosines only of *any* function given in the half Fourier interval; for this reason they are also called the *half-range coefficients* or the cosine and sine coefficients of $f(x)$, respectively.

6·4 Fourier Expansion in $-L,L$ and 0, L

Since in most engineering problems the range of values of the variable x is not limited to the Fourier interval $-\pi,\pi$ or the half Fourier interval $0, \pi$, it is of interest to determine the Fourier expansion of a function $f(x)$ given in any interval $-L,L$ or $0, L$.

To this end we introduce a new independent variable z, which varies in $-\pi,\pi$ as x varies in $-L,L$,

$$z = \frac{\pi}{L}x \qquad (a)$$

and let

$$f(x) = f\left(\frac{L}{\pi}z\right) = \Phi(z) \qquad (b)$$

The new function $\Phi(z)$ defined in $-\pi,\pi$ can now be expanded into a Fourier series by means of Eqs. (6·2·3) and (6·2·4),

$$\Phi(z) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz \qquad (c)$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \Phi(z)\cos nz\,dz \qquad (d)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \Phi(z)\sin nz\,dz \qquad (e)$$

In order to obtain the expansion in terms of the original variable x, substitute Eq. (a) into Eqs. (c), (d), and (e).

$$\Phi(z) = f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \tag{6·4·1}$$

$$a_n = \frac{1}{\pi}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\, d\frac{\pi x}{L} = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\, dx \tag{6·4·2}$$

$$b_n = \frac{1}{\pi}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\, d\frac{\pi x}{L} = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\, dx \tag{6·4·3}$$

Equations (6·4·1) to (6·4·3) give the Fourier expansion of $f(x)$ in any finite interval $-L,L$. This expansion is periodic of period $2L$.

It can be proved that the coefficients a_n and b_n can also be computed by extending the integrals in Eqs. (6·4·2) and (6·4·3) to a complete period $2L$ *starting at any point.* Thus, more in general and with c an arbitrary constant,

$$a_n = \frac{1}{L}\int_{c}^{c+2L} f(x)\cos\frac{n\pi}{L}x\, dx \tag{6·4·2a}$$

$$b_n = \frac{1}{L}\int_{c}^{c+2L} f(x)\sin\frac{n\pi}{L}x\, dx \tag{6·4·3a}$$

For $c = 0$ we obtain the commonly used formulas

$$a_n = \frac{1}{L}\int_{0}^{2L} f(x)\cos\frac{n\pi}{L}x\, dx \tag{6·4·2b}$$

$$b_n = \frac{1}{L}\int_{0}^{2L} f(x)\sin\frac{n\pi}{L}x\, dx \tag{6·4·3b}$$

The half-range expansions in $0, L$ are similarly given by Eq. (6·4·1) with the following coefficients:

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi}{L}x\, dx; \qquad b_n = 0 \qquad \text{in cosine terms} \tag{6·4·4}$$

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi}{L}x\, dx; \qquad a_n = 0 \qquad \text{in sine terms} \tag{6·4·5}$$

For example, the sine expansion of $y = x$ in $(0, 3)$ is given by

$$\begin{aligned} b_n &= \frac{2}{3}\int_0^3 x\sin\frac{n\pi}{3}x\,dx = \frac{2}{n\pi}\int_0^3 x\sin\frac{n\pi}{3}x\, d\frac{n\pi}{3}x \\ &= \frac{2}{n\pi}\left\{-x\cos\frac{n\pi}{3}x\Big]_0^3 + \int_0^3 \cos\frac{n\pi}{3}x\,dx\right\} \\ &= \frac{2}{n\pi}\left\{-3\cos n\pi + \frac{3}{n\pi}\sin\frac{n\pi}{3}x\Big]_0^3\right\} = -\frac{6}{n\pi}\cos n\pi = (-1)^{n-1}\frac{6}{n\pi} \end{aligned}$$

$$f(x) = x = 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin(n\pi/3)x}{n\pi/3} \qquad (0 < x < 3)$$

By means of the half-range coefficients any function given in the interval $(0, L)$ may be expanded into a series containing only cosines or only sines, but a suitable periodic prolongation also allows the expansion of any function given in $(0, L)$ in terms of only odd cosines or only odd sines. To this purpose, notice that, for example, the odd sines have graphs antisymmetrical with respect to the origin 0 and symmetrical with respect to the quarter-period points A,B (Fig. 6·5). If the interval $(0, L)$, in

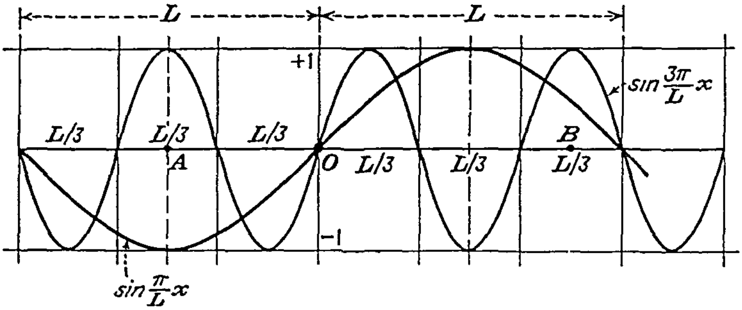


FIG. 6·5.

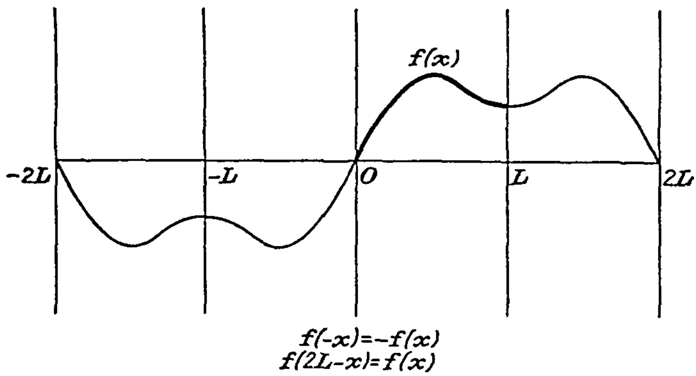


FIG. 6·6.

which $f(x)$ is given, be considered as one-fourth the period $2L$ and the graph of $f(x)$ be prolonged to $-2L$, $+2L$, symmetrically with respect to 0 and antisymmetrically with respect to $-L$ and $+L$ (Fig. 6·6), its expansion will contain only odd sine terms. The coefficients of this expansion are

$$b_n = \frac{2}{2L} \int_0^{2L} f(x) \sin \frac{n\pi}{2L} x \, dx \qquad (n \text{ odd})$$

or, since both $f(x)$ and $\sin (n\pi/2L)x$ (n odd) are symmetrical about $x = L$,

$$b_n = \frac{2}{L} \int_0^{L} f(x) \sin \frac{n\pi}{2L} x \, dx \qquad (n \text{ odd}) \qquad (6·4·6)$$

Similarly, it being noted that the odd cosines are symmetrical about 0 and antisymmetrical with respect to the quarter points, $f(x)$ may be prolonged to $-2L$, $+2L$ in the same manner (Fig. 6·7) and its coefficients will be given by

$$a_n = \frac{2}{2L}\int_0^{2L} f(x)\cos\frac{n\pi}{2L}x\,dx = \frac{2}{L}\int_0^{L} f(x)\cos\frac{n\pi}{2L}x\,dx \quad (n \text{ odd}) \qquad (6\cdot4\cdot7)$$

When the Fourier series expansion of a function contains both cosine and sine terms, these may be combined by noticing that the sum of

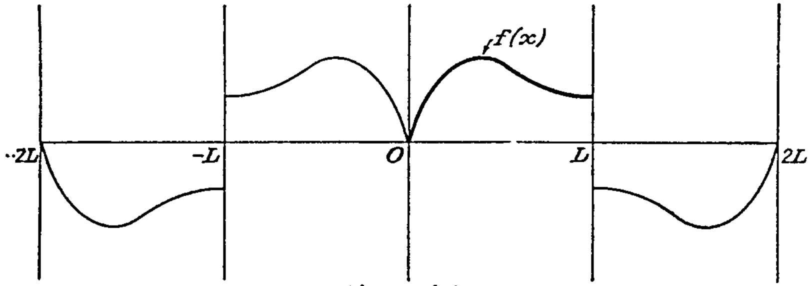


FIG. 6·7.

the nth cosine and the nth sine terms is equivalent to a single cosine term containing an *angle of phase*

$$a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x = c_n \cos\frac{n\pi}{L}(x + x_n) \qquad (f)$$

where c_n is the amplitude and $n\pi x_n/L$ the *phase angle* of the nth harmonic. To determine c_n and x_n, expand the right-hand member of Eq. (f),

$$c_n \cos\frac{n\pi}{L}(x + x_n) = c_n \cos\frac{n\pi}{L}x_n \cos\frac{n\pi}{L}x - c_n \sin\frac{n\pi}{L}x_n \sin\frac{n\pi}{L}x$$

and equate the coefficients of $\cos(n\pi/L)x$ and $\sin(n\pi/L)x$ on both sides of Eq. (f).

$$\left.\begin{aligned} a_n &= c_n \cos\frac{n\pi}{L}x_n \\ b_n &= -c_n \sin\frac{n\pi}{L}x_n \end{aligned}\right\} \qquad (g)$$

Squaring and adding, we obtain

$$c_n = \sqrt{a_n^2 + b_n^2} \qquad (6\cdot4\cdot8)$$

while, taking the ratio, we get

$$\frac{n\pi}{L}x_n = \arctan\frac{-b}{a} \qquad (6\cdot4\cdot9)$$

Of the two angles defined by this equation, that angle must be chosen which satisfies separately both Equations (g).

By means of Eq. (6·4·8) and (6·4·9) the Fourier expansion of $f(x)$ can be written

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi}{L}(x + x_n) \qquad (6\cdot4\cdot10)$$

where the a_n and b_n are given by Eqs. (6·2·10) and (6·2·11).

6·5 Complex Fourier Series

If we substitute in the Fourier expansion of $f(x)$ in the interval $(-\pi,\pi)$ the expressions for $\cos nx$ and $\sin nx$ in terms of imaginary exponentials [Eqs. (b) and (c) of Sec. 5·12),

$$\cos nx = \frac{e^{nix} + e^{-nix}}{2} \qquad \sin nx = \frac{e^{nix} - e^{-nix}}{2i} = -i\,\frac{e^{nix} - e^{-nix}}{2}$$

we obtain

$$\begin{aligned} f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \\ &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \frac{e^{nix} + e^{-nix}}{2} - i \sum_{n=1}^{\infty} b_n \frac{e^{nix} - e^{-nix}}{2} \end{aligned}$$

or, assembling in two separate series all the positive and all the negative exponentials,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \frac{a_n - ib_n}{2} e^{nix} + \sum_{n=1}^{\infty} \frac{a_n + ib_n}{2} e^{-nix} \qquad (a)$$

But since

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

and

$$\cos nx - i\sin nx = e^{-nix} \qquad \cos nx + i\sin nx = e^{nix}$$

the coefficient of e^{nix}, which we shall call c_n, becomes

$$\begin{aligned} c_n = \frac{a_n - ib_n}{2} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)i\sin nx\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx - i\sin nx)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-nix}\,dx \qquad (6\cdot5\cdot1) \end{aligned}$$

Similarly, the coefficient of ϵ^{-nix} becomes

$$\frac{a_n + ib_n}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx + \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)i\sin nx\,dx$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx + i\sin nx)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\epsilon^{nix}\,dx \quad (b)$$

Comparison of Eq. (b) with Eq. (6·5·1) shows that the coefficient $(a_n + ib_n)/2$ is obtained by changing n into $-n$ in the expression for c_n; hence it is logical to let $(a_n + ib_n)/2 = c_{-n}$. Remembering finally that

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

we notice that c_0, that is, the value of Eq. (6·5·1) for $n = 0$, is equal to $\frac{1}{2}a_0$. Hence the Fourier expansion may be written as

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{nix} + \sum_{n=1}^{\infty} c_{-n}e^{-nix} \quad (c)$$

In the second series of Eq. (c), as the running integer n takes the values 1, 2, 3, . . . , the subscript and exponent $-n$ take the values -1, -2, -3, . . . ; hence the series is unaltered if the subscript and exponent $-n$ are changed into $+n$ and the running integer is given the values -1, -2, -3, ...

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{nix} + \sum_{n=-1}^{-\infty} c_n e^{nix}$$

By noticing that $e^{0ix} = 1$, the expansion can finally be written by means of a single sum from $-\infty$ to $+\infty$,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{nix} \quad (6\cdot5\cdot2)$$

where the coefficients c_n are given by Eq. (6·5·1).

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\epsilon^{-nix}\,dx \quad (6\cdot5\cdot1)$$

Equations (6·5·1) and (6·5·2) give the complex form of the Fourier expansion in $(-\pi,\pi)$, which is particularly useful in theoretical derivations. When the expansion is needed in $-L,L$, the corresponding formulas become

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{n\pi i x/L} \tag{6·5·3}$$

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-n\pi i x/L}\, dx \tag{6·5·4}$$

When the expansion is needed in $(0, 2L)$, the corresponding formulas become

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{n\pi i x/L} \tag{6·5·5}$$

$$c_n = \frac{1}{2L} \int_{0}^{2L} f(x) e^{-n\pi i x/L}\, dx \tag{6·5·6}$$

The advantages due to the conciseness and simplicity of the complex Fourier expansion are somewhat impaired by the fact that this form of Fourier series makes no distinction between odd and even functions, *i.e.*, between sine and cosine expansions.

6·6 Dirichlet's Conditions

The conditions for a function to be Fourier expandable are not restrictive. In practice, almost all the functions met in engineering problems are Fourier expandable. The following set of conditions, due to Dirichlet, guarantees that the Fourier expansion of $f(x)$ will converge toward $f(x)$ at all points of continuity:

1. $f(x)$ must never become infinite in the interval of definition.
2. $f(x)$ must be single-valued.
3. $f(x)$ must have, at most, a finite number of discontinuities in the interval of definition.
4. $f(x)$ must have, at most, a finite number of maxima and minima in the interval of definition.

When the periodically prolonged function $f(x)$ is discontinuous, its Fourier coefficients approach zero at least as fast as $1/n$ [see, for example, series (6·2·7)]; when $f(x)$ is continuous with discontinuous first derivative, its coefficients approach zero at least as fast as $1/n^2$ [see, for example, series [6.3.1)]; when $f(x)$ is continuous together with its first $(s - 1)$ derivatives but its sth derivative is discontinuous, its coefficients approach zero at least as fast as $1/n^{s+1}$.

When the prolonged function is continuous, the derivative of its Fourier series gives the Fourier series of its derivative. The integral of the Fourier series of a function equals the Fourier series of the integral of the function.

6·7 Harmonic Analysis

a. In order to investigate the stresses and strains in the structural members of an airplane in flight, special strain gauges are attached to the wings and fuselage of the airplane, and their readings of the strain fluctuations are recorded during the flight.

Many of these records are taken in a complete survey, and it becomes necessary to analyze them in order to determine the causes of the strain variations, since failure in flight is often due to resonance between the natural frequencies of a structural member and the frequencies of the engine disturbances, which are always of a periodic character. The strain records are therefore analyzed for periodic components, as many other experimental periodic functions are analyzed to determine the "spectrum" of their components.

It is proved in the theory of sound that a note emitted by the human voice or by an instrument is a combined vibration in which every component is sinusoidal. The component of lowest frequency is called the *fundamental* or *first harmonic* and gives the pitch of the note; the other components, called the *higher harmonics* of the note, have frequencies that are multiples of the fundamental frequency and define the "color," or "overtones," of the note.

Borrowing the terminology from the theory of sound, the determination of the components of a periodic function is called *harmonic analysis*. Harmonic analysis is generally applied to functions of known period, represented by graphs or tables. Occasionally it is also applied to functions known analytically, whenever it is impossible to perform rigorously the integrations required for the determination of their Fourier coefficients.

Harmonic analysis can be performed by means of special instruments, called *harmonic analyzers*, by geometrical constructions, or by numerical methods. Only numerical methods will be considered here.

b. Given a function $f(x)$ and its Fourier expansion in $(0, 2L)$,

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

in which

$$a_n = \frac{1}{L} \int_0^{2L} f(x) \cos \frac{n\pi}{L} x \, dx$$

$$b_n = \frac{1}{L} \int_0^{2L} f(x) \sin \frac{n\pi}{L} x \, dx$$

an approximate value of the coefficients a_n and b_n can be easily obtained by substituting finite sums for the integrals of a_n and b_n.

Dividing the interval $(0, 2L)$ into N equal parts of length $\Delta x = 2L/N$ and calling $x_r (r = 1, 2, 3, \ldots, N)$ the separation points, we find that

a_n and b_n are given approximately by

$$a_n \doteq \frac{1}{L}\sum_{r=1}^{N} f(x_r)\cos\frac{n\pi}{L}x_r\,\Delta x$$

$$= \frac{2}{N}\sum_{r=1}^{N} f(x_r)\cos\frac{n\pi}{L}x_r \tag{6·7·1}$$

$$b_n \doteq \frac{2}{N}\sum_{r=1}^{N} f(x_r)\sin\frac{n\pi}{L}x_r \tag{6·7·2}$$

The values y_r of the function $f(x)$ at the N points x_r can be taken from the table or the graph of $f(x)$, and the a_n and b_n are twice the average of

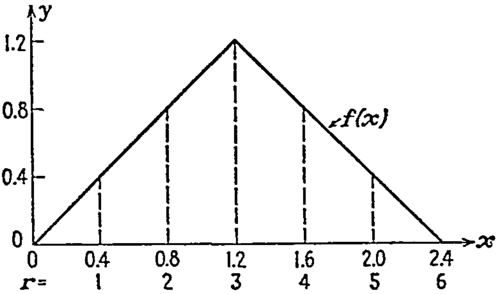


FIG. 6·8.

these values multiplied by the corresponding values of cos $(n\pi/L)x_r$ and sin $(n\pi/L)x_r$, respectively. For the results to be accurate, the number of intervals N into which the period is divided must be high. If $\bar{n}$ is the largest component expected to be present in the graph or table, N must be *at least* equal to $2\bar{n}$ and sometimes much larger than this number.

Table 6·1 shows the computation of a_0,a_1,a_2,a_3,b_1 for the case of a function $f(x)$ in the shape of a triangular wave of period $2L = 2.4$ and amplitude 1.2, taking six subintervals ($N = 6$) (Fig. 6·8). [It is easy to see that all the b_n as well as all the even $a_n(n \neq 0)$ will be zero.]

Although Eqs. (6·7·1) and (6·7·2) give a simple method for the computation of the coefficients a_n and b_n in harmonic analysis, other more practical procedures have been devised and are extensively used.

6·8 The Runge Schemes

Let us assume that the Fourier expansion of $f(x)$ can be stopped after three cosine terms and two sine terms, *i.e.*, that no cosine terms higher than the third and no sine terms higher than the second appear in the graph of $f(x)$. In this case,

$$f(x) = A_0 + A_1\cos\frac{\pi}{L}x + A_2\cos\frac{2\pi}{L}x + A_3\cos\frac{3\pi}{L}x + B_1\sin\frac{\pi}{L}x + B_2\sin\frac{2\pi}{L}x \tag{6·8·1}$$

TABLE 6·1

r	x_r	y_r	$\frac{x_r}{L}\pi$	$\cos\frac{\pi}{L}x_r$	$y_r\cos\frac{\pi}{L}x_r$
1	0.4	0.4	60°	0.5	0.2
2	0.8	0.8	120°	−0.5	−0.4
3	1.2	1.2	180°	−1.0	−1.2
4	1.6	0.8	240°	−0.5	−0.4
5	2.0	0.4	300°	0.5	0.2
6	2.4	0.0	360°	1.0	0.0
		$\Sigma y_r = 3.6$			$\Sigma = -1.6$

r	$\sin\frac{\pi}{L}x_r$	$y_r\sin\frac{\pi}{L}x_r$
1	0.866	−0.3464
2	0.866	0.6928
3	0.000	0.0000
4	−0.866	−0.6928
5	−0.866	−0.3464
6	0.000	0.0000
		$\Sigma = 0.0000$

r	$\frac{2x_r}{L}\pi$	$\cos\frac{2\pi}{L}x_r$	$y_r\cos\frac{2\pi}{L}x_r$
1	120°	−0.5	−0.2
2	240°	−0.5	−0.4
3	360°	1.0	1.2
4	480°	−0.5	−0.4
5	600°	−0.5	−0.2
6	720°	1.0	0.0
			$\Sigma = 0.0$

r	$\frac{3x_r}{L}$	$\cos\frac{3\pi}{L}x_r$	$y_r\cos\frac{3\pi}{L}x_r$
1	180°	−1	−0.4
2	360°	1	0.8
3	540°	−1	−1.2
4	720°	1	0.8
5	900°	−1	−0.4
6	1080°	1	0.0
			$\Sigma = -0.4$

TABLE 6·1 (*continued*)

$$\frac{1}{2}a_0 = \frac{3.6}{6} = 0.6 \qquad a_1 = -\frac{1.6}{3} = -0.533 \qquad a_2 = 0$$

$$a_3 = -\frac{0.4}{3} = -0.133$$

$$b_1 = 0,\ b_2 = 0,\ \ldots,\ b_n = 0$$

$$f(x) = 0.6 - 0.553 \cos\frac{\pi}{1.2}x - 0.133 \cos\frac{3\pi}{1.2}x.$$

The Runge method consists in determining the six coefficients A_0, A_1, A_2, A_3, B_1, and B_2 by compelling the curve (6·8·1) to pass through the six points $x_r, f(x_r)$ $(r = 0, 1, 2, 3, 4, 5)$ of the graph of $f(x)$, whose abscissas divide the period $2L$ into six equal parts. Substituting x_r for x and y_r for $f(x)$ in Eq. (6·8·1), we obtain the following set of six simultaneous equations for the unknown constants A and B:

$$y_r = A_0 + A_1 \cos\frac{\pi}{L}x_r + A_2 \cos\frac{2\pi}{L}x_r + A_3 \cos\frac{3\pi}{L}x_r$$
$$+ B_1 \sin\frac{\pi}{L}x_r + B_2 \sin\frac{2\pi}{L}x_r \quad (r = 0, 1, 2, 3, 4, 5) \qquad (a)$$

It must be noticed that since $x_r = r(2L/6)$, the angles appearing under the sine and cosine functions are of the type

$$\frac{k\pi}{L}x_r = \frac{k\pi}{L}r\frac{2L}{6} = kr\frac{2\pi}{6} \quad (k = 1, 2, 3;\ r = 0, 1, 2, 3, 4, 5)$$

i.e., are all multiplies of $2\pi/6 = 60°$. Hence the coefficients of the unknowns in Eqs. (a) are the numbers ± 1, $\pm\sqrt{3}/2$, $\pm\frac{1}{2}$. Due to this fact, system (a) can be easily solved by means of a simple scheme, called the 6-*ordinate Runge scheme*, in which the known ordinates y_0, y_1, y_2, y_3, y_4, y_5 are added and subtracted and their differences and sums are added and subtracted again as shown in the following table:

6-ORDINATE SCHEME

$$\begin{array}{llll}
 & y_0 & y_1 & y_2 \\
 & y_3 & y_4 & y_5 \\
\hline
\text{Sum} & v_0 & v_1 & v_2 \\
\text{Diff.} & w_0 & w_1 & w_2
\end{array}$$

$$\begin{array}{lll}
v_0 & v_1 & \\
 & v_2 & \\
\hline
p_0 & p_1 & \text{Sum} \\
 & q_1 & \text{Diff.}
\end{array}
\qquad
\begin{array}{ll}
w_0 & w_1 \\
 & w_2 \\
\hline
r_0 & r_1 \\
 & s_1
\end{array}$$

$$\begin{array}{llll}
 & p_0 & & r_0 \\
 & p_1 & & s_1 \\
\hline
\text{Sum} & t_0 & \text{Diff.} & u_0
\end{array}$$

Table 6·2 shows how the coefficients A and B are obtained in terms of the sums and differences of the 6-ordinate scheme. The numbers appearing in the same column are multiplied by the constants appearing in the first column and then added to give three or six times the constants.

TABLE 6·2

Multiplier:						
0.5	....	s_1	$-p_1$	....	....	....
0.866	....	....	....	....	r_1	q_1
1.0	t_0	r_0	p_0	u_0	....	....
	$6A_0$	$3A_1$	$3A_2$	$6A_3$	$3B_1$	$3B_2$

The following example shows the use of the scheme with reference to the triangular wave function of Fig. 6·8:

	0	0.4	0.8
	1.2	0.8	0.4
v	1.2	1.2	1.2
w	−1.2	−0.4	0.4

v_0	v_1	1.2	1.2	w_0	w_1	−1.2	−0.4
	v_2		1.2		w_2		0.4
p_0	p_1	1.2	2.4	r_0	r_1	−1.2	0.0
	q_1		0.0		s_1		−0.8
	p_0	1.2			r_1	−1.2	
	p_1	2.4			s_1	−0.8	
	t_0	3.6			u_0	−0.4	

M	$6A_0$	$3A_1$	$3A_2$	$6A_3$	$3B_1$	$3B_2$
0.5	...	−0.8	−2.4	.....	...	...
0.866	...	.....	.....	.... .	0	0
1.0	3.6	−1.2	1 2	−0.4	...	...
	3.6	−1.6	0	−0.4	0	0

$$A_0 = \frac{3.6}{6} = 0.6 \qquad A_1 = -\frac{1.6}{3} = -0.535 \qquad A_2 = 0$$

$$A_3 = -\frac{0.4}{6} = -0.0667$$

$$B_1 = 0 \qquad B_2 = 0$$

$$f(x) = 0.6 - 0.533 \cos \frac{\pi}{1.2} x - 0.0667 \cos \frac{3\pi}{1.2} x$$

Upon dividing the period $2L$ into 12 parts and taking 6 cosine terms and 5 sine terms, the function

$$f(x) = A_0 + \sum_{n=1}^{6} A_n \cos \frac{n\pi}{L} x + \sum_{n=1}^{5} B_n \sin \frac{n\pi}{L} x \tag{6·8·2}$$

can be made to pass through 12 equally spaced points of the experimental graph or table. The solution of the corresponding system of 12 equations can be obtained by the following 12-*ordinate Runge scheme.*

TWELVE-ORDINATE SCHEME

	y_0	y_1	y_2	y_3	y_4	y_5	y_6
		y_{11}	y_{10}	y_9	y_8	y_7	
Sum	v_0	v_1	v_2	v_3	v_4	v_5	v_6
Diff.		w_1	w_2	w_3	w_4	w_5	

v_0	v_1	v_2	v_3		w_1	w_2	w_3
v_6	v_5	v_4			w_5	w_4	
p_0	p_1	p_2	p_3	Sum	r_1	r_2	r_3
q_0	q_1	q_2		Diff.	s_1	s_2	

	p_0	p_1		r_1	q_0
	p_2	p_3		r_3	q_2
Sum	l_0	l_1	Diff.	t_1	t_2

	p_1	p_0	s_1	l_0
	p_2	p_3	s_2	l_1
Sum	m_1	m_2	u_1	b_1
Diff.	n_1	n_2	z_1	c_1

TABLE 6·3

M	Cosine coefficients						
0.5	....	q_2	n_1	...	$-m_1$	q_2	...
0.866	....	q_1	...	...	...	$-q_1$	...
1.0	b_1	q_0	n_2	t_2	m_2	q_0	c_1
	$12A_0$	$6A_1$	$6A_2$	$6A_3$	$6A_4$	$6A_5$	$12A_6$

M	Sine coefficients				
0.5	r_1	....	....	....	r_1
0.866	r_2	u_1	....	z_1	$-r_2$
1.0	r_3	....	t_1	....	r_3
	$6B_1$	$6B_2$	$6B_3$	$6B_4$	$6B_5$

In the following example of 12-ordinate scheme is applied to the triangular wave of Fig. 6·8.

$$
\begin{array}{c|ccccccc}
 & 0 & 0.2 & 0.4 & 0.6 & 0.8 & 1.0 & 1.2 \\
 & & 0.2 & 0.4 & 0.6 & 0.8 & 1.0 & \\
\hline
v & 0 & 0.4 & 0.8 & 1.2 & 1.6 & 2.0 & 1.2 \\
w & & 0 & 0 & 0 & 0 & 0 &
\end{array}
$$

Since all the w are zero, the r and s and t,u,z are also zero and all the B are zero.

$$
\begin{array}{cccccc}
 & 0 & 0.4 & 0.8 & 1.2 \\
 & 1.2 & 2.0 & 1.6 & \\
\hline
p & 1.2 & 2.4 & 2.4 & 1.2 \\
q & -1.2 & -1.6 & -0.8 &
\end{array}
\qquad
\begin{array}{ccc}
 & 1.2 & 2.4 \\
 & 2.4 & 1.2 \\
\hline
l & 3.6 & 3.6
\end{array}
$$

$$
\begin{array}{cc}
 & -1.2 \\
 & -0.8 \\
\hline
t_2 & -0.4
\end{array}
\qquad
\begin{array}{ccc}
 & 2.4 & 1.2 \\
 & 2.4 & 1.2 \\
\hline
m & 4.8 & 2.4 \\
n & 0 & 0
\end{array}
\qquad
\begin{array}{cc}
 & 3.6 \\
 & 3.6 \\
\hline
b_1 & 7.2 \\
c_1 & 0
\end{array}
$$

M	$12A_0$	$6A_1$	$6A_2$	$6A_3$	$6A_4$	$6A_5$	$12A_6$
0.5	...	−0.8	0	...	−4.8	−0.8	...
0.866	.	−1.60	..	.	.....	1.60	...
1.0	7.2	−1.20	0	−0.4	2.4	−1.2	0
	7.2	−2.9856	0	−0.4	0	−0.2144	0

$$
A_0 = 0.6 \qquad A_1 = -0.4976 \qquad A_2 = 0 \qquad A_3 = -0.0667
$$
$$
A_4 = 0 \qquad A_5 = -0.0357 \qquad A_6 = 0
$$
$$
f(x) = 0.6 - 0.498 \cos \frac{\pi}{1.2} x - 0.0667 \cos \frac{3\pi}{1.2} x - 0.0357 \cos \frac{5\pi}{1.2} x
$$

To check the accuracy of this result we may compare it with the actual Fourier expansion of the same triangular wave,

$$
f(x) = 0.6 - 2.4 \sum_{n=1,3,5,\ldots} \frac{\cos (n\pi/1.2)x}{n^2}
$$

whose first few coefficients are equal to

$$A_0 = 0.6 \qquad A_1 = -0.405 \qquad A_2 = 0 \qquad A_3 = -0.0450$$
$$A_4 = 0 \qquad A_5 = -0.0162 \qquad A_6 = 0$$

Upon dividing the period $2L$ into 24 parts and taking 12 cosine terms and 11 sine terms the function

$$f(x) = A_0 + \sum_{n=1}^{12} A_n \cos \frac{n\pi}{L} x + \sum_{n=1}^{11} B_n \sin \frac{n\pi}{L} x \tag{6·8·3}$$

can be made to pass through 24 equally spaced points of the experimental table or graph. The solution of the corresponding system of 24 equations can be obtained by a scheme similar to the 6- and 12-ordinate schemes.

There have also been worked out 48- and 72-ordinate schemes. These are used in the harmonic analysis of complicated graphs.

A detailed explanation of the 24-ordinate scheme appears in J. B. Scarborough's "Numerical Mathematical Analysis."[1] "Wave Form Analysis"[2] by R. G. Manley contains both the 24- and the 48-ordinate schemes.

6·9 The Selected-ordinate Method

Another numerical method of harmonic analysis, due to Fischer-Hinnen and called the selected-ordinate method, is based upon a particular property of periodic graphs, which we shall now prove.

Given a function $y(x)$ periodic of period $2L$, write its complex Fourier expansion

$$y(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{in\pi x}{L}\right)$$

[exp (x) is a typographically more convenient way of writing e^x when the exponent x is complicated]

Starting from an arbitrary point x_0, divide the period $2L$ into k equal parts by means of the k points

$$x_0, \qquad x_0 + \frac{2L}{k}, \qquad x_0 + 2\left(\frac{2L}{k}\right), \quad \ldots, \qquad x_0 + (k-1)\left(\frac{2L}{k}\right)$$

[1] Johns Hopkins Press, Baltimore, 1930.
[2] John Wiley & Sons, Inc., New York, 1945.

The values of y at these points are given by the series

$$\left.\begin{aligned}
y(x_0) &= \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{in\pi x_0}{L}\right) \\
y\left(x_0 + \frac{2L}{k}\right) &= \sum_{n=-\infty}^{\infty} c_n \exp\left[\frac{in\pi}{L}\left(x_0 + \frac{2L}{k}\right)\right] \\
&= \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{in\pi x_0}{L}\right) \exp\left(\frac{in2\pi}{k}\right) \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
y\left[x_0 + (k-1)\left(\frac{2L}{k}\right)\right] &= \sum_{n=-\infty}^{\infty} c_n \exp\left\{\frac{in\pi}{L}\left[x_0 + (k-1)\left(\frac{2L}{k}\right)\right]\right\} \\
&= \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{in\pi x_0}{L}\right) \exp\left(in\,\frac{k-1}{k}\,2\pi\right)
\end{aligned}\right\} \tag{a}$$

Indicating with $\sum_{0}^{k-1} y_k$ the sum of the k values (a) and letting

$$a = e^{in2\pi/k} \tag{b}$$

we obtain

$$\sum_{0}^{k-1} y_k = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{in\pi x_0}{L}\right)[1 + e^{in2\pi/k} + e^{2in2\pi/k} + e^{3in2\pi/k} + \cdots + e^{(k-1)(in2\pi/k)}]$$

$$= \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{in\pi x_0}{L}\right)(1 + a + a^2 + a^3 + \cdots + a^{k-1}) \tag{c}$$

or, since

$$1 + a + a^2 + \cdots + a^{k-1} = \frac{1 - a^k}{1 - a}$$

(see Sec. 5·10 a)

$$\sum_{0}^{k-1} y_k = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x_0/L}\,\frac{1 - e^{in2\pi}}{1 - e^{i2\pi n/k}} \tag{d}$$

We now notice that the exponential

$$e^{im2\pi} = \cos m2\pi + i \sin m2\pi$$

equals 1 when m is an integer and is different from 1 when m is not an integer. Hence, whenever n/k is not an integer, $e^{i2\pi n/k}$ is different from 1 and the denominators of the fractions in Eq. (d) are different from zero. The numerators of these fractions are, instead, equal to zero, since $e^{in2\pi} = 1$; therefore *all the terms of the series* (d) *for which* n/k *is not an integer, i.e., for which* n *is not a multiple of* k, *vanish.*

The terms for which n/k is an integer, say r, assume the indeterminate form 0/0, but in this case $a = e^{ir2\pi} = 1$ and the sum of the terms in the parentheses of Eq. (c) is k.

In conclusion, the sum (d) contains only those terms for which n is a multiple of k, $n = rk$, and it may be written as

$$\sum_{0}^{k-1} y_k = k \sum_{r=-\infty}^{\infty} c_{rk} \exp\left(\frac{irk\pi x_0}{L}\right) \tag{e}$$

The complex expansion (e) may be transformed into a trigonometric expansion by reversing the procedure used in Sec. 6·2,

$$\sum_{r=-\infty}^{\infty} c_{rk} \exp\left(\frac{irk\pi x_0}{L}\right) = \sum_{r=-\infty}^{-1} c_{rk} \exp\left(\frac{irk\pi x_0}{L}\right) + c_0$$
$$+ \sum_{r=1}^{\infty} c_{rk} \exp\left(\frac{irk\pi x_0}{L}\right) = c_0 + \sum_{r=1}^{\infty} \left[c_{rk} \exp\left(\frac{irk\pi x_0}{L}\right)\right.$$
$$\left. + c_{-rk} \exp\left(\frac{-irk\pi x_0}{L}\right)\right] = c_0 + \sum_{r=1}^{\infty} \left(A_{rk} \cos\frac{rk\pi x_0}{L} + B_{rk} \sin\frac{rk\pi x_0}{L}\right)$$

and hence Eq. (e) becomes

$$\sum_{0}^{k-1} y_k = kc_0 + k\sum_{r=1}^{\infty} \left(A_{rk} \cos\frac{rk\pi x_0}{L} + B_{rk} \sin\frac{rk\pi x_0}{L}\right) \tag{6·9·1}$$

in which

$$c_0 = \frac{1}{2L}\int_0^{2L} y(x)\,dx = \text{average ordinate of the } y \text{ graph} \tag{6·9·2}$$

Equation (6·9·1) is the basis of the Fischer-Hinnen method.

If in Eq. (6·9·1) we let $x_0 = 0$, that is, *if the period* $2L$ *is divided into* k *equal parts starting from its left end* and we call $A^{(k)}$ the corresponding sum of ordinates,

$$\sum_{0}^{k-1} y_k \Big]_{x_0=0} = A^{(k)} \tag{6·9·3}$$

we obtain

$$A^{(k)} = kc_0 + k\sum_{r=1}^{\infty} A_{rk} = kc_0 + k(A_k + A_{2k} + A_{3k} + \cdots)$$

from which

$$A_k = \frac{1}{k}A^{(k)} - c_0 - (A_{2k} + A_{3k} + A_{4k} + \cdots) \tag{6·9·4}$$

If in Eq. (6·9·1) we let $x_0 = 2L/4k$, that is, *divide $2L$ into $4k$ parts and consider the ordinates $y_1, y_5, y_9, y_{13}, \ldots$*, we obtain

$$\cos\frac{rk\pi x_0}{L} = \cos\frac{rk\pi L}{L2k} = \cos r\frac{\pi}{2} = \begin{cases} 0 \text{ for } r \text{ odd} \\ -1 \text{ for } r = 2m \text{ and } m \text{ odd} \\ 1 \text{ for } r = 2m \text{ and } m \text{ even} \end{cases}$$

$$\sin\frac{rk\pi x_0}{L} = \sin\frac{rk\pi L}{L2k} = \sin r\frac{\pi}{2} = \begin{cases} 0 \text{ for } r \text{ even} \\ 1 \text{ for } r = 2m+1 \text{ and } m \text{ even} \\ -1 \text{ for } r = 2m+1 \text{ and } n \text{ odd} \end{cases}$$

and, upon calling

$$\sum_{0}^{k-1} y_k\Big]_{x_0=L/2k} = B^{(k)}$$

Eq. (6·9·1) becomes

$$B^{(k)} = kc_0 + k(-A_{2k} + A_{4k} - A_{6k} + A_{8k} - \cdots + B_k - B_{3k} + B_{5k} - B_{7k} + \cdots)$$

from which

$$B_k = \frac{1}{k}B^{(k)} - c_0 + (A_{2k} - A_{4k} + A_{6k} - A_{8k} + \cdots + B_{3k} - B_{5k} + B_{7k} - B_{9k} + \cdots) \tag{6·9·5}$$

The evaluation of the coefficients (6·9·4) and (6·9·5) is started by computing the average ordinate c_0; this is done by evaluating the area under the y graph by Simpson's rule (see Sec. 1·9 *f*) and dividing it by the period $2L$.

The index k_m of the *highest* harmonic appearing in the graph is then estimated and A_{k_m} and B_{k_m} are computed by means of Eqs. (6·9·4) and (6·9·5), in which all the A and B appearing in the right-hand members are neglected, since their indices are multiples of k_m and k_m is the highest index considered.

$$A_{k_m} = \frac{1}{k_m}A^{(k_m)} - c_0$$

$$B_{k_m} = \frac{1}{k_m}B^{(k_m)} - c_0$$

The next lower index k is then considered, and the corresponding A_k and B_k are computed by means of Eqs. (6·9·4) and (6·9·5), the procedure being continued until A_1 and B_1 are evaluated.

The method will now be applied to the triangular wave of Fig. 6·8. The area under the y graph equals $(2.4 \times 1.2)/2 = \overline{1.2}^2$, and the average ordinate is $c_0 = \overline{1.2}^2/2.4 = 0.6$. The highest harmonic considered will be the third. Dividing the period 2.4 into $k = 3$ equal parts, starting at $x = 0$, we obtain

$$A^{(3)} = 0 + 0.8 + 0.8 = 1.6$$

and

$$A_3 = \tfrac{1}{3} \times 1.6 - 0.6 = -0.0667$$

Dividing the period 2.4 into $4k = 12$ parts and considering the ordinates $y_1, y_5, y_9, \ldots$, we obtain

$$B^{(3)} = 0.2 + 1.0 + 0.6 = 1.8$$

and

$$B_3 = \tfrac{1}{3} \times 1.8 - 0.6 = 0$$

Similarly, dividing the period into $k = 2$ equal parts, starting at $x = 0$,

$$A^{(2)} = 0 + 1.2 = 1.2$$

$$A_2 = \frac{1.2}{2} - 0.6 = 0$$

and, dividing the period into $4k = 8$ equal parts,

$$B^{(2)} = 0.3 + 0.9 = 1.2$$

$$B_2 = \frac{1.2}{2} - 0.6 = 0$$

Dividing the period into $k = 1$ parts,

$$A^{(1)} = 0$$

$$A_1 = \frac{1}{k} A^{(1)} - c_0 - (A_2 + A_3 + A_4 + \cdots)$$

$$= 0 - 0.6 - (0 - 0.0667) = -0.5333$$

and, dividing the period into $4k = 4$ parts,

$$B^{(1)} = 0.6$$

$$B_1 = \frac{1}{k} B^{(1)} - c_0 + (A_2 - A_4 + \cdots) + (B_3 - B_5 + \cdots)$$

$$= \frac{1}{1} \times 0.6 - 0.6 + 0 + 0 = 0$$

The values of the constants thus obtained are, in this case, identical with the values obtained by the 6-ordinate scheme.

When the highest harmonic to be considered is the 6th, we obtain, similarly,

$$
\begin{aligned}
k &= 6 \qquad & A^{(6)} &= 0 + 0.4 + 0.8 + 1.2 + 0.8 + 0.4 = 3.6 \\
& & A_6 &= \frac{3.6}{6} - 0.6 = 0 \\
k &= 5 & A^{(5)} &= 0 + 0.48 + 0.96 + 0.96 + 0.48 = 2.88 \\
& & A_5 &= \frac{2.88}{5} - 0.6 = -0.024 \\
k &= 4 & A^{(4)} &= 0 + 0.6 + 1.2 + 0.6 = 2.4 \\
& & A_4 &= \frac{2.4}{4} - 0.6 = 0 \\
k &= 3 & A^{(3)} &= 0 + 0.8 + 0.8 = 1.6 \\
& & A_3 &= \frac{1.6}{3} - 0.6 - A_6 = -0.0667 \\
k &= 2 & A^{(2)} &= 0 + 1.2 = 1.2 \\
& & A_2 &= \frac{1.2}{2} - 0.6 - (A_4 + A_6) = 0 \\
k &= 1 & A^{(1)} &= 0 \\
& & A_1 &= \frac{0}{1} - 0.6 - (A_2 + A_3 + A_4 + A_5 + A_6) \\
& & &= -0.6 - 0 + 0.0667 - 0 + 0.024 - 0 \\
& & &= -0.5093
\end{aligned}
$$

The B have not been computed since they are equal to zero.

These results are different from those obtained from a 12-ordinate scheme, which shows that the two procedures are not equivalent.

It should be noted that the fundamental formulas (6·9·4) and (6·9·5) of the selected-ordinate method have been obtained by chosing $x_0 = 0$. Hence the graph to be analyzed must always be referred to an x axis with origin at the left end of the graph.

Problems

1. Prove the orthogonality conditions

$$
\begin{aligned}
&(a) \quad \int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0 \\
&(b) \quad \int_0^{2\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}
\end{aligned}
\qquad (m,n = \text{integers})
$$

2. (a) What essential conditions should be satisfied by a family of functions $f_1(x), f_2(x), \ldots, f_n(x), \ldots$ in order that the functions may be used in expansions of the Fourier type with interval (a,b)?

(b) Why would a series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cosh nx + \sum_{n=1}^{\infty} b_n \sinh nx$$

not yield as fruitful results as a Fourier series?

3. Determine the constants $A_1, A_2, A_3, B_2, B_3, C_3$ such that the polynomials

$$\begin{aligned} f_1(x) &= A_1 \\ f_2(x) &= A_2 + B_2 x \\ f_3(x) &= A_3 + B_3 x + C_3 x^2 \end{aligned}$$

form an orthogonal set in the interval (0, 1) with

$$\int_0^1 \overline{f_i(x)}^2 \, dx = 1 \qquad (i = 1, 2, 3)$$

4. (a) Expand in a Fourier series the function illustrated in Fig. 6·9.

(b) Sketch the series using 1, 2, and 3 terms of the expansion.

(c) What would be the expansion of the function if the x axis were shifted up half a unit?

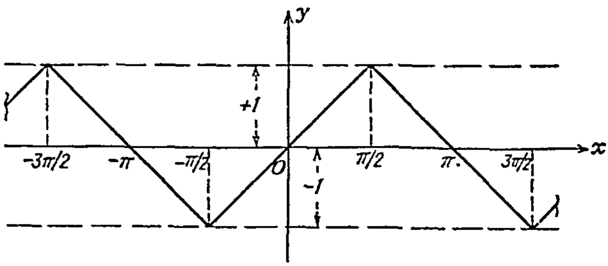


Fig. 6·9.

5. Expand in a Fourier series the function illustrated in Fig. 6·10. What is the value of the series at 0? at π?

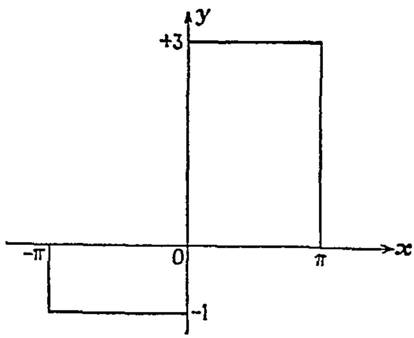


Fig. 6·10.

6. Expand in a Fourier series the function defined in the complete Fourier interval by the sketch of Fig. 6·11.

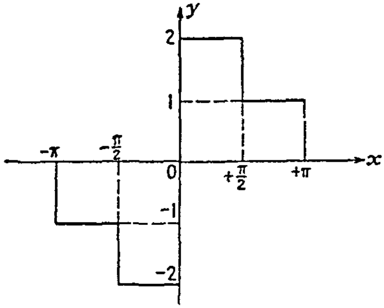


Fig. 6·11.

7. Sketch a square wave ($y = -\pi$, $-\pi < x < 0$; $y = \pi$, $0 < x < \pi$) with the first and third harmonics removed.

8. Given a Fourier expandable function in an interval (x_1,x_2), is it always possible to obtain (*a*) a Fourier series containing only sine terms, (*b*) a Fourier series containing only cosine terms, that will represent the given function in the interval (x_1,x_2)? Discuss.

9. The function shown in Fig. 6·12 is defined in the half Fourier interval. Expand it (*a*) in sine terms only, (*b*) in cosine terms only.

10. Expand the function shown in Fig. 6·13 as an even function.

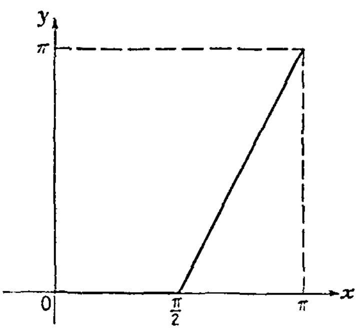


Fig. 6·12.

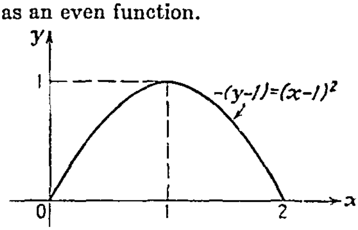


Fig. 6·13.

11. Expand the function defined in the complete Fourier interval $(-2, +2)$ in Fig. 6·14. What is the value of the series at 0? at $+2$?

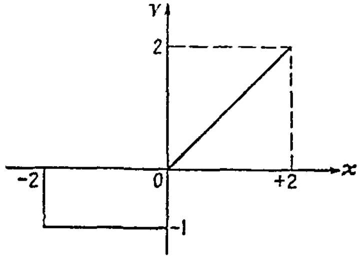


Fig. 6·14.

12. Expand the function defined in the complete Fourier interval $(-1, +1)$ in Fig. 6·15. What is the value of the series at 0?

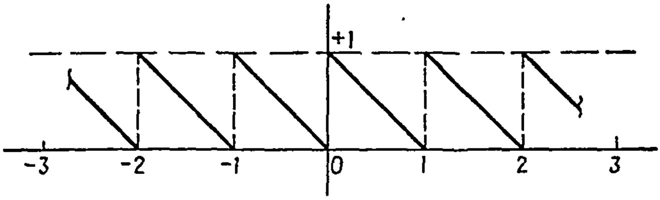


FIG. 6·15.

13. Expand the function shown in Fig. 6·16

(a) in sine terms only
(b) in cosine terms only

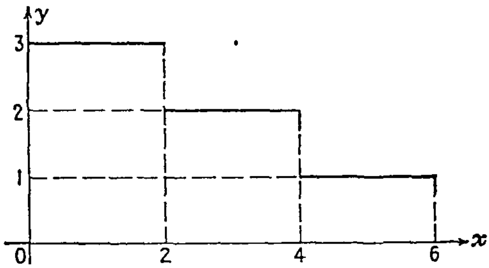


FIG. 6·16.

14. Expand the function shown in Fig. 6·17

(a) in sine terms only
(b) in cosine terms only

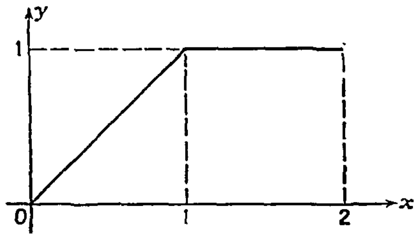


FIG. 6·17.

15. (a) Expand the function shown in Fig. 6·18 in a Fourier series.

(b) Taking the limit as a approaches zero in the coefficients of the previous expansion, obtain the expansion of the so-called "unit-impulse" function.

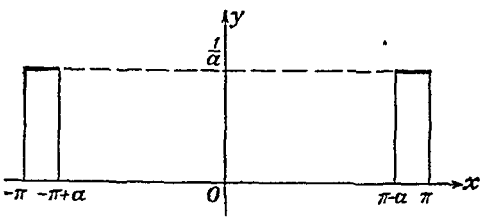


FIG. 6·18.

Note: While this latter series is not convergent, many useful engineering problems can be solved by its means.

16. Write the series expansion of the function shown in Fig. 6·19 in the form

$$y = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi}{L}x + \alpha_n\right)$$

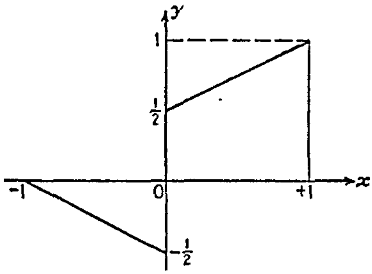


FIG. 6·19.

17. Expand in a complex Fourier series the function $y = e^x$ defined in the interval $(-1, 1)$.

18. A voltage-wave shape has the form given in Fig. 6·20.

(*a*) By referring $e(t)$ to a new set of axes is it possible to transform $e(t)$ into an even or an odd function?

(*b*) What is the fundamental frequency of $e(t)$?

(*c*) What is the period of its third harmonic?

(*d*) Calculate the first term ($\frac{1}{2}a_0$) of the Fourier expansion of $e(t)$, that is, the so-called "d-c component."

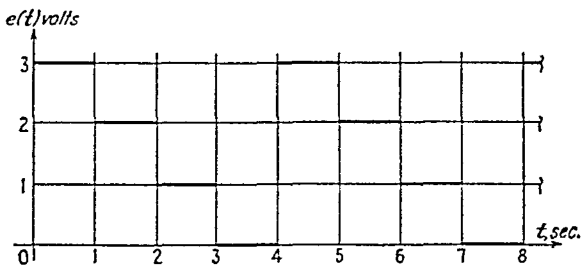


FIG. 6·20.

19. Expand in a Fourier sèries the function obtained by passing $y = \sin x$ through a half-wave rectifier. *Hint:* A half-wave rectifier clips off the negative portions of the input wave.

20. What are the characteristic properties of a function whose Fourier expansion contains only

(*a*) even cosine terms
(*b*) odd cosine terms
(*c*) even sine terms
(*d*) odd sine terms

21. Which of the following functions may and which may not be expanded into Fourier series in the intervals indicated?

(a) $\sin \dfrac{1}{2x}$ $\quad (-\pi,\pi)$

(b) $\dfrac{1}{(x-1)^2}$ $\quad (0, 2)$

(c) $y^2 = 1 - x^2$ $\quad (-1, 1)$

(d) $\dfrac{1}{x+1}$ $\quad (-0.1, 0.1)$

22. How fast will the Fourier coefficients of the following functions approach zero with increasing n?

(a) $f(x)$ of Prob. 4
(b) $f(x)$ of Prob. 10
(c) $f(x)$ of Prob. 14
(d) y of Prob. 17

23. (a) At what points does the derivative of the Fourier series of the functions of Prob. 22a to c give values different from the derivatives of the functions?

(b) Does the integrated Fourier series of the same functions give the Fourier series of their integrals?

24. Apply the 6-ordinate Runge scheme to the following functions in the intervals indicated:

(a) $y = x$ $\quad (0, 6)$
(b) $y = x^2$ $\quad (0, 3)$
(c) $y = \cosh x$ $\quad (-3, 3)$
(d) $y = \log_{10} x$ $\quad (1, 4)$

25. Apply the 12-ordinate Runge scheme to the following functions: (a) the function of Fig. 6·21

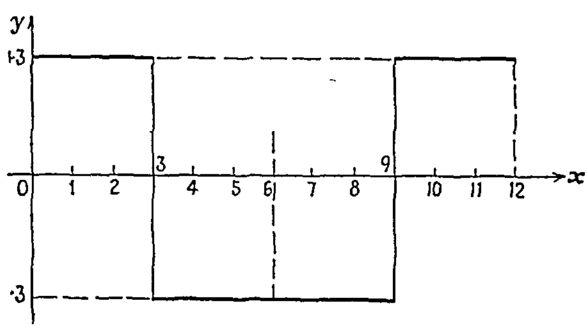


Fig. 6·21.

Note: Take $y = 0$ at points of discontinuity.

(b)

x	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0
y	1	1.2	1.5	1.9	2.3	2.0	1.7	1.4	1.0	0.6	0.8	0.9	1.0

26. Analyze by the selected-ordinate method the following functions in the intervals given up to the indicated harmonic:

(a) $y = (x - 1)^2$ $\quad (0 < x < 2)$ $\quad$ 4th harmonic

(b) $y = \begin{cases} 1 - x & (0 < x < 1) \\ x - 1 & (1 < x < 2) \end{cases}$ $\quad$ 4th harmonic

(c) $y = \begin{cases} -2 & (0 < x < 2) \\ +2 & (2 < x < 4) \end{cases}$ $\quad$ 6th harmonic

Note: Take $y = 0$ at points of discontinuity.

(d) $y = x$ $\quad (0 < x < 3)$ $\quad$ 6th harmonic

Note: Analyze $y = x - 3/2$, and add $3/2$ to the result. Why?

ANSWERS TO ALTERNATE PROBLEMS

Chapter I

1. A rational number is an integer or a fraction. An irrational is a real number neither an integer nor a fraction. An imaginary is a number whose unit is i. A complex number is a number composed of a certain number of real units and a certain number of imaginary units.
3. $\sqrt{4}$ is the diagonal of a square, whose side ($\sqrt{2}$) can be built as the diagonal of a square of side 1. $\sqrt{10}$ is the diagonal of a square, whose side ($\sqrt{5}$) can be built as the hypotenuse of a right triangle of sides 1 and 2.
5. Assume $\log_{10} 5 = m/n$, with m,n integers without common factors. Hence $10^{m/n} = 5$, or $2^m 5^m = 5^n$, that is, $2^m = 5^{n-m}$. But 2^m is even, and 5^{n-m} is odd. Hence the assumption $\log_{10} 5 = m/n$ is false and $\log_{10} 5$ is an irrational number.
6. (*b*) Fraction; (*d*) integer; (*f*) complex; (*h*) fraction; (*j*) fraction
7. (*b*) 4.5; (*d*) $\sqrt{13}$
8. (*b*) $\frac{3}{4}(1 - \sqrt{3}i)$; (*d*) $-\frac{1}{5}(1 + 3i)$; (*f*) $\frac{3}{4} - \frac{1}{2}i$; (*h*) $46 + 9i$; (*j*) $\frac{1}{2809}(-106 + 371i) = -0.0377 + 0.132i$
9. (*b*) $1 + 6i$; (*d*) $a(3 + 4a + 6i)$; (*f*) $\frac{2}{25}(3 + 4i)$; (*h*) $-8 + 12i$; (*j*) $5 + 12i$
10. (*b*) $-2 + 4i$; (*d*) -4; (*f*) $+\sqrt{2}\,i$
11. (*b*) $7.280\underline{/16°}$; (*d*) $0.001802\underline{/303°40'}$; (*f*) $5.10\underline{/281°20'}$; (*h*) $0.01\underline{/270°}$; (*j*) $2.04 \times 10^6\underline{/348°40'}$
12. (*b*) $0.299 - 0.193i$; (*d*) $5.36 + 4.50i$; (*f*) $1.90 + 1.20i$
13. (*b*) $2.01\underline{/20°50'}$; (*d*) $2.35 \times 10^{-4}\underline{/164°}$; (*f*) $r^2\underline{/0°}$; (*h*) $r^2\underline{/\frac{\pi}{2}}$
14. (*b*) $2.41,\ -1.21 + 2.09i,\ -1.21 - 2.09i$; (*d*) $1.85 + 0.765i,\ -1.85 - 0.765i,\ -0.765 + 1.85i,\ +0.765 - 1.85i$
    (*f*) $2.11\underline{/0.2, 1.77, 3.34, 4.91}$; (*h*) $\pm(1.516 - 4.62i)$
    (*j*) $+0.257 + 0.703i,\ -0.590 + 0.461i,\ -0.620 - 0.417i,\ +0.206 - 0.720i,\ +0.748 - 0.0268i$
15. (*b*) $\pm 1.189,\ \pm 1.189i$
    (*d*) $\pm(1.225 - 1.225i)$
    (*f*) $0.866 + 0.5i,\ i,\ -0.866 + 0.5i,\ -0.866 - 0.5i,\ -i,\ 0.866 - 0.5i$
19. $z = 20\underline{/22.5°} = 18.48 + 7.65i$
21. (*a*) Height of a tree, temperature in a room, speed of a car, sales of stock at the New York Stock Exchange
    (*c*) Volume of a parallelepiped, distance between two points in space
    (*e*) Amount of work done by a man, amount of money spent in running a refrigerator
22. (*b*) $x = 4 \sin \frac{2\pi}{7} t$
24. (*a*) $|x| \le 6$; (*c*) $x > 0$; (*e*) $|x| \le 3$; (*g*) $2n\pi < x < (2n + 1)\pi$
25. (1) Velocity of a body as a function of the coordinate, the coordinate being a function of time

(2) Volume of a gas depending on the pressure, function of the temperature
(3) Cost of manufacture depending on later costs, which depend on number of manufactured items

27. x must approach x_0, x must remain near x_0

28. (b) Zero; (d) no limit; (f) 1; (h) $+\infty$; (j) zero

29. (b) $\frac{4}{3}$; (d) zero; (f) zero

30. (b) None; (d) $\frac{1}{2}$; (f) $\frac{\pi}{2}, \frac{3\pi}{2}$; (h) 1

31. (b) $\pm\infty$; (d) 1; (f) $\frac{1}{\sqrt{2}}$

32. (b) $x^{2/3}$

33. (b) First; (d) not an infinitesimal; (f) order $\frac{1}{5}$;
(h) first; (j) first

34. (b) $6x^2 - 7$; (d) $\cos 2x$; (f) nx^{n-1}; (h) $\frac{1}{\cos^2 x}$

35. (b) knx^{n-1}; (d) $\frac{1}{\sqrt{2x}}$; (f) $(b - 7x)6x - 7(a + 3x^2)$;
(h) $-\sqrt{1 + x^2} + \frac{(2 - x)x}{\sqrt{1 + x^2}}$; (j) $x^{n-1}(1 + x^n)^{(1-n)/n}$

36. (b) $\frac{2}{x}$; (d) $\cot x$; (f) $\frac{2}{\sin 2x} + \frac{3}{x}$; (h) $\cos x\, e^{\sin x}$; (j) $\frac{3e^{3x}}{\cos^2 e^{3x}}$

37. (b) 1; (d) $e^{-x}\left(\frac{1 - \sin x \cos x}{\cos^2 x}\right)$; (f) $b^{4x}\left(4 \ln b \ln 4x + \frac{1}{x}\right)$; (j) $\frac{1}{\ln x^x}$

38. (b) $-\frac{1}{(1 + x^2)(\tan^{-1} x)^2}$; (d) $\sec^{-1} x + \frac{1}{\sqrt{x^2 - 1}}$; (f) $-\frac{1}{e^x + e^{-x}}$;
(h) $-\frac{1}{x\sqrt{1 - x^2}}$; (j) $\frac{6}{\sinh 2x}$

39. (b) $\cosh x\, e^{\sinh x}$; (d) $-e^{-2x}$; (f) $\csc^{-1} x$; (h) zero;
(j) $\frac{5}{\cos^2 (x/2)} \frac{1}{16 + [5 \tan (x/2) + 3]^2}$

40. (b) $\frac{2 \sin x}{\cos^3 x}$; (d) $-3 \cos^2 x \sin x$; (f) $\sin x \sinh x + \cos x \cosh x$;
(h) $-\frac{\sin x}{\cos^2 \sin^2 (\sec x - 4)}$; (j) $\frac{28x^6}{3} \sqrt[3]{\sin x^7}$

41. (b) $2 \sin x + x \cos x$; (d) $-\frac{4 \cos x}{(x - 1)^2} [(x^2 - 1)(\sin x + \cos x)]$
(f) $\frac{3x^2}{\sin x^3}\left(1 - \frac{x^3}{\sin x^3}\right)$; (h) $\frac{2x^2 \tan x^2 - 1}{x\sqrt{x^2 \cos^2 x^2 - 1}}$
(j) $2 \cos x^2 - \frac{1}{x^2} \sin x^2$

42. (b) $-\sin t$; (d) $\frac{6}{(144t - 143)\sqrt{t - 1}}$; (f) $\frac{1}{t(\ln t + 3)}$
(h) $\frac{2}{\sqrt{1 - t^2}} [\cos (2 \sin^{-1} t)]$; (j) $\frac{(1 + \sin t)}{\cos t}$

44. (b) $3x^2 + 3y$, $3y^2 + 3x$, $6x$, $6y$, 3
(d) $2 \cos (2x + y)$, $\cos (2x + y)$, $-4 \sin (2x + y)$, $-\sin (2x + y)$, $-2 \sin (2x + y)$
(f) $\frac{1}{x}, \frac{1}{y}, -\frac{1}{x^2}, -\frac{1}{y^2}, 0$

45. (b) $12x$; (d) $-2 \sin 2x$; (f) $n(n-1)x^{n-2}$; (h) $\dfrac{2 \sin x}{\cos^3 x}$

48. $\sin x$, $\cos x$

50. $z = 4 + y + c$

52. $\dfrac{dx}{dt} = -\alpha x$

54. $\dfrac{\partial(1/r)}{\partial x} = -\dfrac{x-a}{r^3}$, $\dfrac{\partial^2(1/r)}{\partial x^2} = -\dfrac{1}{r^3} + \dfrac{3(x-a)^2}{r^5}$

56. (a) 10.15; (c) 7.047; (e) 0.004926; (g) 0.9657; (i) 0.0198

57. 0.43 per cent

58. (a) 1.05 cu ft per sec; (b) 0.482 sq ft per sec

60. $d(1+\epsilon)^m \Big]_{\epsilon=0} = m(1+\epsilon)^{m-1}\Big]_{\epsilon=0} d\epsilon = m\, d\epsilon$

$d(1+m\epsilon)\Big]_{\epsilon=0} = m\, d\epsilon$

$(1+\epsilon)^m (1+\eta)^n = (1+m\epsilon)(1+n\eta) = 1 + m\epsilon + n\eta + mn\epsilon\eta$
$\doteq 1 + m\epsilon + n\eta$

or use binomial expansion

61. (b) zero; (d) zero; (f) 1; (h) e^{ab}; (j) ∞; (l) $1/24$

62. (b) $-\frac{1}{4}\cos 4x + c$; (d) $\frac{1}{2}\sinh^2 x + c$; (f) $\ln x - 4/x + c$

(h) $-\sqrt{a^2 - x^2} + c$; (j) $\frac{1}{2}x\,[\sin(\ln x) - \cos(\ln x)] + c$

63. (b) $\tan x + c$; (d) $\ln(x-1) + c$; (f) $\dfrac{-1}{b}\cos(a+bx) + c$

(h) $\dfrac{\sqrt{a}}{3}\sqrt{(2x-a)^3} + c$; (j) $-\dfrac{2(2a-bx)}{3b^2}\sqrt{a+bx} + c$

64. (b) 4.045; (d) $\pi (m = n)$, zero $(m \neq n)$; (f) 0.014; (h) -4.40 (j) 0.363

65. (b) $\frac{1}{2}e^x(x \sin x - x\cos x + \cos x) + c$

(d) $\dfrac{e^{2x}}{10}(\cos 4x + 2 \sin 4x) + c$

(f) $\dfrac{e^{3x}}{6}(\sin 3x - \cos 3x)$

66. (b) $\dfrac{1}{\omega^2}(\omega t - \sin \omega t)$; (d) $\dfrac{1}{\omega}t^2 + \dfrac{2}{\omega^3}(\cos \omega t - 1)$

67. $\displaystyle\int_0^\infty x^n e^{-x}\,dx = -\int_0^\infty x^n\,de^{-x} = -\left\{x^n e^{-x}\Big]_0^\infty - n\int_0^\infty x^{n-1}e^{-x}\,dx\right\}$

$\displaystyle = n\int_0^\infty x^{n-1}e^{-x}\,dx = n(n-1)\int_0^\infty x^{n-2}e^{-x}\,dx = \cdots = n!$

68. (b) 2.004571, 2.000283; $e_2 = 0.00032$; (d) 28.70; (f) 2.609

69. 20.2×10^{-6} coulombs

71. 3.180

73. $A_c = \dfrac{4}{3}\dfrac{c^3}{k} = A_P = \dfrac{2c^3}{k} - \displaystyle\int_{-c/k}^{c/k}(c^2 - y)\,dx$

75. 0.821 amp

77. (a) $\sqrt{2/3}E$, $3/4E$; (b) $\sqrt{2/3}E$, 0

79. $\dfrac{k}{(1-r)}\left[v_2^{(1-r)} - v_1^{(1-r)}\right]$

81. Mep = 114.3 psi

83. $P = \frac{2}{5}\gamma H$

85. $H = \dfrac{2kI}{h}$

87. $V = \frac{4}{3}a^3$
89. $m = \frac{2}{3}ka^3$
91. The integral of an odd function in the interval $(-a, +a)$ is zero; the integral of *an even function is unchanged by changing x into $-x$.*
93. 1700 ft
96. $Pcr = \dfrac{\pi^2 EI}{(\sqrt{\frac{2}{3}}L)^2}$
98. 20 Btu per hr

Chapter II

1. (a) 7.81, 1.2; (c) 1.41, -1; (e) 13.60, -3.25; (g) 1.12, -5.5; (i) 1.41, -1
2. (a) $y = 0.5x + 2$; (c) $x = 0$
3. (a) $4y + 3x - 22 = 0$, $m = -\frac{3}{4}$, $b_x = 7.33$, $b_y = 5.5$
   (c) $2y + 3x + 2 = 0$, $m = -\frac{3}{2}$, $b_x = -\frac{2}{3}$, $b_y = -1$
   (e) $4y - 3x - 15 = 0$, $b_x = -5$, $b_y = 3.75$
4. (a) $m = -\frac{3}{2}$, $b_x = 1.33$, $b_y = 2$
   (c) $m = -4$, $b_x = 0$, $b_y = 0$
   (f) $m = 0.388$, $b_x = -4.85$, $b_y = 1.88$
5. (a) $(-3, 2.5)$; (c) $(0,0)$
6. (a) $2y + x - 2 = 0$, $y - 2x - 1 = 0$
   (c) $3y - x + 16.2 = 0$, $y + 3x - 1.6 = 0$
   (e) $x = 0$, $y = 0$
   (g) $y - x - 1.14 = 0$, $y + x - 1.14 = 0$
7. (a) $\theta = 30°$; (c) 90°
8. (a) 0.671; (c) 4.28; (e) 24; (g) 806
9. Coordinates of mid-points of sides: $\left(\frac{b}{2}, \frac{c}{2}\right)$, $\left(\frac{a+b}{2}, \frac{c}{2}\right)$; $l = \frac{a}{2}$
11. 1
13. (4.46, 10.19), (-2.46, -0.196)
15. 5.00
17. The first automobile, by 23.5 min
18. (b) $(x + 2.4)^2 + (y - 1.2)^2 = 23.04$
    (d) $(x - 1.1)^2 + (y - 4.2)^2 = 68.00$
    (f) $(x - 1.33)^2 + (y - 2.54)^2 = 8.22$
19. (b) (0, 0.75) $r = 1.031$
    (d) (1.19, -0.194) $r = 1.41$
20. A circle
22. 37°
23. (b) $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$
24. (b) $\dfrac{x^2}{4} - \dfrac{y^2}{2.25} = 1$
    (d) $\dfrac{y^2}{3} - \dfrac{x^2}{1} = 1$, $\dfrac{y^2}{2.25} - \dfrac{x^2}{4} = 1$, $\dfrac{y^2}{16} - \dfrac{x^2}{9} = 1$
25. (b) $y = \pm 0.803x$; (d) $x = 0$, $y = 0$
26. (b) $y = -2x^2$; (d) $y = 1.48x^2 - 1.44x$
28. Length of side $= \dfrac{\sqrt{3}}{2}$
31. (a) Ellipse; (c) circle; (e) parabola
32. (a) $x^2 + y^2 = 4$; (c) $x^2 - y^2 = 9$; (e) $y = \sin \dfrac{2x^2}{9}$

33. (*a*) $(x')^2 + (y')^2 = 4$; (*c*) $x' + y' = 0$
34. (*b*) $2.21x^2 + 0.793y^2 = 2$; (*d*) $y' = -\sqrt{2}$
37. $\theta = 45°$, $x_0 = 1.18$, $y_0 = -0.707$
39. $x_0 = 3/8$, $y_0 = -2/7$
41. $\bar{x} = 1/5$, $\bar{y} = 9/5$
42. (*b*) $r = \cos\theta$
 (*d*) $r^2(1 - 3\sin^2\theta) = 8$
 (*f*) $r^2[b^2 - (a^2 + b^2)\sin^2\theta] = a^2b^2$
 (*h*) $f = \dfrac{\sin\theta}{a^2\cos^2\theta}$
44. $I^2 = \displaystyle\int_0^\infty \int_0^{\pi/2} re^{-r^2}\,dr\,d\theta = \frac{\pi}{4}$
45. (*b*) $y = 1$; (*d*) $y = 3.46x + 3.46$
46. (*b*) $y = -2.718x$; (*d*) $y = \pm 0.577(x - 4)$
48. $2y^2 = (x - 1)$
49. (*b*) Decreasing; (*d*) increasing; (*f*) increasing
50. (*b*) -0.0377; (*d*) -0.00260; (*f*) -0.0949
52. $C = \pm \dfrac{a}{b^2}$
54. $C = +1$
55. (*b*) Negative; (*d*) positive; (*f*) negative
56. (*b*) Minimum at $x = +\sqrt{7/3}$; maximum at $x = -\sqrt{7/3}$, inflection at $x = 0$
 (*d*) Maximum at $x = 0$; minimum at $x = \pm 0.5$;
 inflection points at $x = \pm 0.289$
 (*f*) Inflection point at $x = 1$
57. (*b*) $\sqrt{2}\,a$, $\dfrac{a}{\sqrt{2}}$
58. Depth 2.5 in., width 5 in.
61. 60°, 60°, 60°, that is, equilateral
63. 60 stories
65. $s = \dfrac{1}{2}\dfrac{v_0^2}{g}$
67. (250 × 500) sq ft
69. $d = 500$ miles
71. $x_0 = 25.3$ ft
73. 4 in. × 4 in. × 2 in.
75. $\alpha = 45°$
77. $\theta = \dfrac{\pi}{4}$
79. $x = 90.5$ ft
81. 0.558 mile from stronger light
83. $v = c$
86. 6 floors, 4 lots
88. $x = 0.519L$
89. (*b*) $l = 24$ in.; $r = \dfrac{24}{\pi} = 7.67$ in.
91. $X_L = -X_g$, $R_g = R_L$
93. $t = 0$ $(x = -1, y = 10)$
95. $R = \dfrac{\pi\rho}{D}\dfrac{1}{\ln(A/a)}$

97. 1.28 hr

99. $\frac{S\sqrt{2}}{2}$

Chapter III

1. (a) $x = \frac{1}{3}$; (c) $x = \frac{4a^2b}{5b - 5a^2 + ab^2}$; (e) $x = \frac{7ab}{7b - 3}$
2. $x = 1 - \frac{\ln (a + b)}{\ln a}$
4. $V = 128.5$ ft per min
6. 185.7 per cent
8. $22.2x$
10. $C = 2$ knots
11. (b) $x = -1 \pm \sqrt{2}\,i$; (d) $x = -1.165$, $x = +0.436$; (f) $x = 3, 3$
13. $x^2 - 8x + 25 = 0$, $x = 4 \pm 3i$
15. $t = 1.95$ sec
17. $V_A = 31.30$ mph, $V_B = 21.30$ mph
19. $x = 125$, $C = \$20$
21. $x = 2$
22. (b) $\sqrt{2}\,(\pm 1 \pm i)$; (d) $\pm\sqrt{3}$, $\pm i$; (f) ± 4.08, $\pm 1.26i$
24. (b) $0 < x < 1$; (d) $-4 < x < -3$;
    (f) $-4 < x < -3$, $-1 < x < 0$, $4 < x < 5$
    (h) $-1 < x < 0$, $1 < x < 2$; (j) $0 < x < 0.9$ and $x = 1$
25. (c) $x = 1.2$; (g) $x = -1.5$
26. (a) $x = -2.76$; (c) $x = 1.21$; (e) $x = 0.453$
27. (g) -1.54, $1.16 \pm 0.669i$
28. (b) $+4, +3$; (d) $-2, -7$; (f) $0, +5$; (h) $0, -1$; (j) $0, +2$
29. (b) Positive, 3 or 1; negative, 0; imaginary, 2 or 0
    (d) Positive, 2 or 0; negative, 1; imaginary, 2 or 0
    (f) Positive, 1; negative, 2 or 0; imaginary, 2 or 0
    (h) Positive, 1; negative, 1; imaginary, 2
    (j) Positive, 2 or 0; negative, 0; imaginary, 4,2
32. $a = 3V_c^2P_c$; $b = \frac{V_c}{3}$; $R = \frac{8P_cV_c}{3T_c}$
34. $t = 0.256$ ft
36. $\alpha = 24°06'$
38. $f = 0.00890$
40. (a) $x = 0.244$; (c) $x = 0.588$
41. $\sqrt{r^2 - 1} + r^2 \sin^{-1}\left(\frac{1}{r}\right) = 4$, $r = 2.1$
43. $t = 1.83RC$
45. $n = 4$

Chapter IV

1. (a) $y = \frac{-8}{2a + 3b}$, $x = \frac{5b + 6a}{2a + 3b}$
   (c) $x = 1$, $y = 1$
   (e) $x = \frac{5}{3} + \frac{4}{3}y$; any y
3. (a) $x = 1$, $y = 2$, $z = 3$
   (c) $x = 4$, $y = -1$, $z = -2$
12. (a) 729; (c) -1203

14. (a) $x_1 = 1,\ x_2 = 1,\ x_3 = -1,\ x_4 = 2$
(c) $x_1 = 1.25,\ x_2 = -1.42,\ x_3 = 2.12,\ x_4 = -2.12,\ x_5 = 3.07$

16. $x = \dfrac{a}{\Delta}(1 - \cos\alpha\cos 3\alpha - 3\sin\alpha\sin 3\alpha)$

$y = \dfrac{a}{3\Delta}(3 - 3\cos\alpha\cos 3\alpha - \sin\alpha\sin 3\alpha)$

$z = \dfrac{a}{\Delta}(3\sin 3\alpha\cos\alpha - \cos 3\alpha\sin\alpha)$

$t = \dfrac{a}{3\Delta}(\cos 3\alpha\sin\alpha - 3\sin 3\alpha\cos\alpha)$

where $\Delta = 4(5\sin\alpha\sin 3\alpha + 3\cos\alpha\cos 3\alpha - 3)$

17. (b) $x_1 = 1,\ x_2 = 2,\ x_3 = 3,\ x_4 = 4$
18. (b) $x_1 = 2,\ x_2 = -1,\ x_3 = -1$
20. $x_1 = 1.20,\ x_2 = 1.70,\ x_3 = 1.45$
21. (b) Inconsistent
23. $\dfrac{x}{z} = -1,\ \dfrac{y}{z} = -3$
25. $x = -20/9$
27. $z_{12} = \dfrac{Rz_{12}}{R + z_{22}}$
28. (b) $I_1 = 9.24^a,\ I_2 = 5.19^a,\ I_3 = 11.51^a$
29. $3x^2 + 2xy - 4x + 7y - 7 = 0$
30. (b) $M_1 = +\dfrac{qL^2}{28},\ M_2 = -\dfrac{qL^2}{14}$
(d) $M_1 = -12.5$ ft-kip, $M_2 = -31.3$ ft-kip, $M_3 = -73.5$ ft-kip, $M_4 = -64.2$ ft-kip, $M_5 = -12.5$ ft-kip

Chapter V

1. (a) 20, 12; (c) 499, 3; (e) 12.62, −4.2
2. (a) $-\dfrac{3}{8}\dfrac{1}{s+1} + \dfrac{1}{4}\dfrac{1}{s-1} + \dfrac{1}{8}\dfrac{1}{s-3}$

(c) $\dfrac{1}{2}s + \dfrac{3}{8}\dfrac{1}{s - (1/\sqrt{2})} + \dfrac{3}{8}\dfrac{1}{s + (1/\sqrt{2})}$

(e) $\dfrac{1}{2}\dfrac{1}{s+1} + \dfrac{1}{2}\dfrac{s}{s^2 - s + 2}$

(g) $\dfrac{1}{s+2} - \dfrac{s}{s^2 + 2s + 1}$

(i) $\dfrac{3}{4}\dfrac{s-1}{s^2+1} - \dfrac{3}{4}\dfrac{s-3}{s^2 - 2s - 1}$

6. (a) −0.904; (c) −15.90; (e) +0.584
7. (a) $-\sin\theta$; (c) $\cot\theta$; (e) $\sin\theta$; (g) $\csc\theta$; (i) $\cos\theta$
8. (a) 0.9690; (c) 2.28; (e) −4.84
9. (a) Neither; (c) odd; (e) odd; (g) even
14. (a) 54°, 3.07; (c) 82°50′, 0.185; (e) 48°50′, 1.19
15. Odd; neither; odd; neither
16. (b) 6351; (d) 7.436; (f) 0.4895
17. (b) $\sqrt{x}$; (d) $\dfrac{1}{x^2}$; (f) $\dfrac{x^2}{y}$; (h) $\dfrac{1}{x^y}$
18. (b) 0.295; (d) 1.34; (f) 0.994; (h) −4.11
19. (b) $x > 3.8$

28. $i = I \sin \omega t + \frac{mI}{2}[\cos(\omega_2 - \omega)t - \cos(\omega_2 + \omega)t]$

37. $y = -\frac{c}{24}\csc^2\alpha,\ x = \frac{c}{8}(\cot^2\alpha - 1),\ t = +\frac{c}{12}\cot^3\alpha$

$z = -\frac{c}{4}\cot^4\alpha$

42. 50°35′ east of north

44. $x = 8.87$ yd

46. $A = 33{,}200$ sq ft

48. $P = 11.633$ million

50. (a) $1 + \frac{1}{2}x - \frac{1}{8}x^2$

(c) $a^{2/3} - \frac{1}{3}\frac{bx}{a^{1/3}} - \frac{1}{9}\frac{b^2x^2}{a^{4/3}}$

(e) $1 - nx + \frac{n(n-1)}{2}x^2$

51. (b) 3.107; (d) 1.974; (f) 4.023

52. (b) 0.523; (d) 1.049; (f) 0.427

53. (b) $\cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \cdots$

(d) $e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \cdots$

(f) $\arcsin x = x + \frac{x^3}{6} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \cdots$

(h) $\sqrt{1 - x^2} = 1 - \frac{x^2}{2} - \frac{1}{8}x^4 - \cdots$

54. (b) $\frac{x^6}{6!}\cosh x$

(g) $\frac{-6}{(1 + x)^4}\frac{x^4}{4!}$

55. (b) $e^x = e\left[1 + (x - 1) + \frac{(x - 1)^2}{2!} + \cdots\right]$

(d) $\ln x^2 = \ln 4 + (x - 2) - \frac{1}{4}(x - 2)^2 + \cdots$

(f) $\tan x = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \cdots$

56. (b) $\frac{e^x}{3!}(x - 1)^3$; (d) $\frac{2}{3}\frac{1}{x^3}\frac{(x - 2)^3}{3!}$

57. (b) 0.0175; (d) 1.120; (f) 11.12

59. $\arctan x$

61. 1.6858

62. (b) 0.271; (d) 0.272; (f) 1.312

65. $\Delta = r\left(\frac{\theta^3}{3} - \frac{\theta^5}{60}\right)$; $\theta < 14°$

67. $x = r\cos\omega t - l\left[1 - \frac{1}{2}\left(\frac{r}{l}\right)^2\sin^2\omega t\right]$

$v = -\omega r\left(\sin\omega t - \frac{1}{2}\frac{r}{l}\sin 2\omega t\right)$

69. $y = A(x - x_0)^4 + B$

70. (b) Divergent; (d) divergent; (f) convergent; (h) convergent
(j) convergent

72. (a) Convergent; (c) divergent; (e) divergent; (g) convergent

73. (b) $-1 < x < +1$; (d) $-\infty < x < \infty$; (f) $-1 \leq x < +1$

(h) $-\frac{1}{3} \leq x \leq \frac{1}{3}$, $\frac{3^n x^n}{n^2 + 1} = n$th term

(j) Always divergent

(l) $2 \leq x \leq 4$

75. (a) $-1 + 42i$; (c) $\frac{\sqrt{2}}{2}(1 - i)$; (d) $-0.757i$; (g) $11.54i$

(i) $-11.54i$; (k) $-4i$

76. (a) $x = a$, no limit; $x = bi$, zero

77. (a) 1; (c) $|\sin x|$; (e) 1.12

79. (b) $x = 0$, $x = \pm 2.646$

$y = \pm 2.646i$, $y = 0$

(d) $x = 0.2$, $y = 0.4$

78. (b) $x = \frac{\pi}{2} i$

(d) $x = \frac{\pi}{2n}$

Chapter VI

2. (a) $\int_a^b \varphi_i(x)\varphi_j(x)\,dx = \begin{cases} A = \text{const}; i = j \\ 0; i \neq j \end{cases}$

(b) The sinh and cosh functions are neither periodic nor orthogonal

4. (a) $y = \frac{8}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin nx}{n^2}$

5. $y = 1 + \frac{8}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin nx}{n}$

6. $y = \frac{6}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin nx}{n} + \frac{4}{\pi} \sum_{n=2,6,10,\ldots}^{\infty} \frac{\sin nx}{n}$

8. (a) Yes; (b) yes

10. $y = \frac{2}{3} - \frac{16}{\pi^2} \sum_{n=2,4,6,\ldots}^{\infty} \frac{\cos(n\pi/2)}{n^2}$

12. $y = \frac{1}{2} + \frac{2}{\pi} \sum_{n=2,4,6,\ldots}^{\infty} \frac{\sin n\pi x}{n}$

$y(0) = \frac{1}{2}$

14. (a) $y = \frac{4}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2}(-1)^{(n+3)/2} \sin \frac{n\pi x}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n}(-1)^n \sin \frac{n\pi x}{2}$

16. $f(x) = \frac{1}{4} - \frac{2}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sqrt{1 + \pi^2 n^2}}{n^2} \cos(n\pi x + \tan^{-1} n\pi)$

18. (a) Yes. New Origin $(4, \frac{3}{2})$. Odd

(c) $T = \frac{4}{3}$ sec

20. (b) $f(x) = f(-x)$; $f(x - L) = -f(x)$. A function even in the complete Fourier interval; odd in each half interval

(d) $f(-x) = -f(x)$; $f(x - L) = -f(x)$. A function odd in the complete Fourier interval and even in each half interval

22. (b) At least as $\frac{1}{n^2}$

24. (b) $y = 2.292 - 0.500 \cos \frac{2\pi x}{3} - 1.167 \cos \frac{4\pi x}{3} - 0.625 \cos \frac{6\pi x}{3}$

$+ 2.598 \sin \frac{2\pi x}{3} - 0.866 \sin \frac{4\pi x}{3}$

(d) $y = 0.3161 - 0.1423 \cos \frac{2\pi x}{3} - 0.1171 \cos \frac{4\pi x}{3} - 0.0567 \cos \frac{6\pi x}{3}$

$- 0.1570 \sin \frac{2\pi x}{3} - 0.0554 \sin \frac{4\pi x}{3}$

26. (b) $y = \frac{1}{2} + 0.444 \cos \pi x + 0.0556 \cos 3\pi x$

(d) $y = 1.5 - 0.850 \sin \frac{2\pi x}{3} - 0.500 \sin \frac{4\pi x}{3} - 0.250 \sin \frac{6\pi x}{3}$

$- 0.1875 \sin \frac{8\pi x}{3} - 0.150 \sin \frac{10\pi x}{3} - 0.125 \sin \frac{12\pi x}{3}$

INDEX